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ON NORM ATTAINING OPERATORS ACTING FROM $L^1(\mu)$ TO $C(S)$

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An operator acting between two Banach spaces X and Y is called norm attaining if there exists an x in the unit ball of X such that

$$\|Tx\| = \|T\| .$$

The classical result of Bishop and Phelps asserts that if Y is one-dimensional then the norm attaining operators (functionals) are dense in the Banach space $L(X,Y)$. Lindenstrauss [3] proved the same assertion for X reflexive and Y arbitrary. The interesting case of $X = L^1[0,1]$ was analyzed by Uhl [5]. Several results for $Y = C(S)$, the space of continuous functions on a compact Hausdorff space S, were obtained by Johnson and Wolfe [1]. The pair $X = L^1(\mu)$, $Y = C(S)$ fits to none of the general results. The problem whether the norm attaining operators are dense in $L(L^1[0,1], C[0,1])$ was formulated in [1], recently a negative solution has been announced in [2].

Here we present positive approximation results for certain operators in $L(X,Y)$ where $X = L^1(\mu)$ for some finite measure μ and $Y = C(S)$ with S compact Hausdorff. We discuss two classes of such operators defined in terms of continuity in measure (Section 1) and L^∞-separability (Section 2). Both classes contain the weakly compact operators. In Section 4 we slightly modify Lindenstrauss' result ([3], Theorem 1) to prove the denseness of operators with norm attaining adjoints in arbitrary Banach spaces.

Recall that for any T $L(L^1(\mu), C(S))$ the mapping $s \longrightarrow T^*\delta_s$ is a weak* continuous function from S to $L^\infty(\mu)$ and this correspondence defines an isometric isomorphism between $L(L^1(\mu), C(S))$ and the space of weak* continuous functions from S to $L^\infty(\mu)$ with the supremum norm. Compact operators correspond to the norm continuous functions and weakly compact operators correspond to the weakly continuous functions. Clearly T is norm attaining iff for some $s \in S$

$$|T^*\delta_s(x)| = \|T\|$$

on a non-null set B. The equality $|Tf(s)| = \|T\|$ must then hold for every normalized function f supported in B and satisfying on B the condition $\operatorname{sign} f = \operatorname{sign} T^*\delta_s$.

1. Continuity in measure. The map $s \longrightarrow T^*\delta_s$ can be viewed as a weakly continuous function from S into $L^1(\mu)$. In the case of $S = [0,1]$, by Ryff [4], the set of L^1- continuity points is co-meager. We consider the following stronger condition:

(*) The mapping $s \longrightarrow T^*\delta_s$ is continuous in measure.

Note that all (Bochner) representable operators satisfy (*). Indeed, if T is represented by a bounded strongly measurable function g, i.e. $Tf = \int fg\, d\mu$ for $f \in L^1(\mu)$, then the range of g, so the range of T, is separable and contained in a separable closed subalgebra A of C(S). By the Gelfand representation, A is isomorphic to some $C(\bar{S})$ with $\bar{S}$ compact and metrizable. The mapping $s \longrightarrow T^*\delta_s$ factors through $\bar{S}$, so it suffices to prove the sequential continuity in (*). Now, by the dominated convergence theorem, for every sequence (s_n) converging to s in S

$$\int f(x)(g(x)(s_n) - g(x)(s))\, d\mu(x) \longrightarrow 0$$

uniformly in $\|f\|_\infty \leq 1$. This implies $g(\cdot)(s_n) \longrightarrow g(\cdot)(s)$ in measure, so (*) holds in view of the equality $g(\cdot)(s) = T^*\delta_s$ in $L^\infty(\mu)$.

To see that (*) need not imply Bochner representability, it suffices to consider the operator $T \in L(L^1[0,1], C[0,1])$ determined by $s \longrightarrow \chi_{[0,s]}$.

By the Dunford-Pettis-Phillips theorem all weakly compact operators in $L(L^1(\mu), C(S))$ are Bochner representable, hence satisfy (*). The argument in [5], Theorem 1, shows directly that representable operators in $L(L^1(\mu), Y)$ can be approximated by norm attaining representable operators. It is not hard to see that if the representable operator happens to be weakly compact then an approximating operator can also be chosen weakly compact. (Another proof of the fact that the weakly compact operators can be approximated by norm attaining weakly compact operators in $L(L^1(\mu), C(S))$ is an easy consequence of Theorem 2 in Section 2.) Here we have the following observation.

Theorem 1. Every operator satisfying (*) can be approximated by norm attaining operators satisfying (*).

Proof. Denote $h_s = T^*\delta_s$. For $0 < \varepsilon < \|T\|$ we define

$$\bar{h}_s = (h_s \wedge (\|T\| - \varepsilon)) \vee (-\|T\| + \varepsilon) .$$

Since the lattice operations are continuous in L^1- norm, $s \longrightarrow \bar{h}_s$ is continuous in measure. In particular $\bar{h}_s$ is weak* continuous and represents an operator $\bar{T} \in L(L^1(\mu), C(S))$ which clearly satisfies $\|T - \bar{T}\| < \varepsilon$. To show that $\bar{T}$ is norm attaining note that for some $s_0 \in S$

$$|h_{s_0}(x)| \geq \|T\| - \varepsilon$$

on a non-null set C. This implies $|\bar{T}^*\delta_{s_0}(x)| = |\bar{h}_{s_0}(x)| = \|T\| - \varepsilon = \|\bar{T}\|$ on C.

2. Separability condition. Consider the operators in $L(L^1(\mu), C(S))$ determined by the condition:

(**) There exists a co-meager set $G \subset S$ such that $\{T^*\delta_s : s \in G\}$ is norm separable in $L^\infty(\mu)$.

If S has a countable dense subset of isolated points then (**) is automatically satisfied for every $T \in L(L^1(\mu), C(S))$. (In this special case the denseness of norm attaining operators follows from Proposition 3 in [3].)

Note that all weakly compact operators satisfy (**). Indeed, by the Dunford-Pettis-Phillips theorem every weakly compact operator T in $L(L^1(\mu), Y)$ has separable range, hence the range of the weakly compact adjoint T^* is also separable.

Theorem 2. Every operator satisfying (**) can be approximated by norm attaining operators satisfying (**).

Proof. Let $h_s = T^*\delta_s$ and let $h^1, h^2, \ldots$ be a norm dense sequence in the closure of $\{h_s : s \in G\}$. The sets

$$\{x : a < h^k(x) < b\},$$

where a,b run over the rational numbers and $k \geq 1$, form a countable family $B_1, B_2, \ldots$ We define

$$v_n(s) = \operatorname*{ess\,inf}_{x \in B_n} h_s(x)$$

$$w_n(s) = \operatorname*{ess\,sup}_{x \in B_n} h_s(x).$$

The v_k's are upper semi-continuous and the w_k's are lower semi-continuous, so the set F of the continuity points of all these functions is co-meager. For every $\varepsilon > 0$ the set

$$\{ s : \|h_s\| > \|T\| - \varepsilon/3 \}$$

is nonempty and open ($\|h_s\|$ is l.s.c.) so there exists $s_0 \in F \cap G$ with $\|h_{s_0}\| > \|T\| - \varepsilon/3$. Without loss of generality suppose

$$h_{s_0}(x) > \|T\| - \varepsilon/3$$

on a non-null set. There exists an h^k such that $\|h^k - h_{s_0}\| < \varepsilon/3$ hence $h^k(x) > \|T\| - 2\varepsilon/3$ on a non-null set. The inequality must hold on a non-null B_n so $v_n(s_0) > \|T\| - \varepsilon$. By the choice of s_0, we have $v_n(s) > \|T\| - \varepsilon$ on a neighborhood U of s_0. Now choose a continuous function $0 \leq r(s) \leq 1$ with $r(s_0) = 1$ and $r(s) = 0$ for $s \in S \setminus U$. Define

$$k_s(x) = h_s(x) + \chi_{B_n}(x) r(s)(\|T\| - h_s(x)).$$

Clearly $s \longrightarrow k_s$ is weak* continuous and approximates h_s within ε. It is easily seen that k_s represents an operator T_0 satisfying (**) and attaining its norm at every normalized function $f \geqslant 0$ supported in B_n.

It should be noted that by letting $r(s) = 1$ in a neighborhood of s_0 in the above proof we obtain a norm attaining operator T_0 which approximates T and attains its norm "strongly" in the following sense:

(+) There exist a non-null B and a nonempty open set $V \subset S$ such that $|T_0 f(s)| = \|T_0\|$ for every $s \in V$ and every nonnegative normalized function f supported in B.

3. Examples. The operators satisfying (+) are not dense in the set of all norm attaining operators in $L(L^1[0,1], C[0,1])$. In fact we shall construct a positive norm attaining operator T which satisfies (*) and cannot be approximated by operators satisfying (+).

Given $\alpha > 0$ and $0 \leqslant t \leqslant 1$ we define an L^1- continuous mapping $s \longrightarrow k_s$ from $[0,1]$ into $L^\infty[0,1]$. First, for every $s \neq t$ let

$$a(s) = |\sin(1/(t - s))|$$

$$b(s) = \alpha |s - t| / \max(t, 1 - t)$$

and define $I(s) = (a(s), (a(s) + b(s)) \wedge 1)$ for $s \neq t$, $I(t) = \emptyset$. Clearly $|I(s)| \leqslant \alpha$, $s \longrightarrow k_s = 1 - \chi_{I(s)}$ is L^1- continuous, and $\|k_s\| = 1$. Moreover, in every neighborhood U of t and for every set B of positive measure $k_s(x) = 0$ holds on a non-null subset of B for some $s \in U$. Now take a sequence (α_n) of positive numbers with $\sum \alpha_n < 1$ and a countable dense subset $\{t_1, t_2, \ldots\}$ in (0,1). Denote by $s \longrightarrow k_s^n$ the L^1- continuous mapping constructed as above for α_n and t_n. The functions $s \longrightarrow k_s^1 \wedge \ldots \wedge k_s^n$ are still L^1- continuous and for every s converge in L^1-norm to a mapping $s \longrightarrow h_s$. Due to $\sum \alpha_n < \infty$ the convergence is uniform in s, so $s \longrightarrow h_s$ is continuous in measure. Clearly the corresponding operator is positive of norm 1, norm attaining, and satisfies (*). For every nontrivial rectangle of the form $B \times V$ the representing function h_s vanishes on a non-null subset of B for some s in V. This shows that T cannot be approximated by operators satisfying (+).

It should be pointed out that Bochner representable operators in $L(L^1(\mu), C(S))$ can always be approximated by operators satisfying (+). This follows essentially from the approximation by countably valued functions (cf. the proof of Theorem 1 in [5]). On the other hand, representable operators need not satisfy (**). Indeed, consider $T \in L(L^1(\mu), C(\{0,1\}^N))$ given by $Tf(s) = \int fs \, d\mu$ where μ is a strictly positive finite measure on the set of natural numbers N. Here T is represented by the projection mapping $\pi_n \in C(\{0,1\}^N)$ on N while every uncountable subset of $\{0,1\}^N = \{T^*\delta_s : s \in \{0,1\}^N\}$ is nonseparable in $L^\infty(\mu)$.

4. Norm attaining adjoints. The adjoint T^* of $T \in L(L^1(\mu), C(S))$ attains its norm iff $\|T^*\delta_s\| = \|T\|$ for some s. Since the function $s \longrightarrow \|h_s\| = \|T^*\delta_s\|$ is l.s.c., the set of its continuity points is dense. Choose such a point s_0 with $\|h_{s_0}\|$ close to $\|T\|$ and then multiply h_s by a continuous scalar function r satisfying $r(s_0) = \|T\|/\|h_{s_0}\|$. If r is appropriately chosen, $s \longrightarrow r(s)h_s$ will represent an operator $R \in L(L^1(\mu), C(S))$ with the adjoint attaining its norm at δ_{s_0} and with $\|R - T\|$ small. This shows that every operator can be approximated by operators with norm attaining adjoints. The following theorem, closely modeled on Lindenstrauss' argument [3], gives the same for arbitrary Banach spaces. Since T^{**} attains its norm whenever T^* does (and the converse is not true in general) this is a slightly stronger version of Theorem 1 in [3].

Theorem 3. Let E, F be arbitrary Banach spaces. The operators with norm attaining adjoints are dense in L(E,F).

Proof. Let $R \in L(E,F)$ with $\|R\| = 1$ and $0 < \varepsilon < 1/3$ be given. As in [3] we choose a decreasing sequence of positive numbers $\varepsilon_1, \varepsilon_2, \ldots$ such that $2\sum \varepsilon_k < \varepsilon$ and $2\sum_{i>k} \varepsilon_i < \varepsilon_k^2$ for $k \geqslant 1$. Note that $\varepsilon_k < 1/6$. In order to reduce our notation to that in [3] we let $E^* = Y$, $F^* = X$, and $R^* = T$. Now choose sequences (T_k)

in $L(X,Y)$, (x_k) in X, and (f_k) in $E \subset Y^*$ satisfying

(a) $T_1 = T$

(b) $\|T_k x_k\| \geqslant \|T_k\| - \varepsilon_k^2$, $\|x_k\| = 1$

(c) $f_k(T_k x_k) \geqslant \|T_k x_k\| - \varepsilon_k^2$, $\|f_k\| = 1$

(d) $T_{k+1}x = T_k x + \varepsilon_k f_k(T_k x) T_k x_k$.

Note that in contrast to [3] we cannot require that $f_k(T_k x_k) = \|T_k x_k\|$ in (c) because of the restrictive condition $f_k \in E$. The additional ε_k^2, however, decreases so rapidly that the following argument remains unaltered. Repeating along the lines the proof in [3] (with the same auxiliary inequalities (8) - (11)) we obtain $\lim T_k = \hat{T}$, $\|\hat{T} - T\| \leqslant \varepsilon$, and $|f_j(\hat{T}x_k)| \geqslant \|\hat{T}\| - \delta_j$ for $k > j$, where $\delta_j = 6\varepsilon_j + 2\varepsilon_{j-1}^2$. Each T has a pre-adjoint R_k in $L(E,F)$ since $f_k \in E$. The adjoint operators form a closed subspace, so $\hat{T} = \hat{R}^*$ for some $\hat{R} \in L(E,F)$. Clearly $\|\hat{R} - R\| \leqslant \varepsilon$ and $\hat{R}^*$ attains its norm on every weak* limit point of the sequence (x_k).

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