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ON ISOMETRIC DOMAINS OF POSITIVE OPERATORS ON ORLICZ SPACES

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The purpose of this note is to establish a characterization of L^p-spaces, where $1 \le p < +\infty$, in class of Orlicz spaces in terms of positive operators acting on them.

Given real Banach space E, we denote by $\mathcal{L}(E)$ the Banach space of all bounded linear operators from E into E. For an operator $T \in \mathcal{L}(E)$ we define its isometric domain M(T) as

$$\{ f \in E : \|Tf\| = \|T\| \, \|f\| \}$$

(see [2]) .

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a convex strictly increasing function with $\varphi(0) = 0$. We denote by L_φ the corresponding Orlicz space equipped with the norm $\|\cdot\|$, sometimes called the Luxemburg norm of L_φ. That is, L_φ is the linear space of all equivalence classes of Lebesgue measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that $\int \varphi(|f(x)|/\alpha)\, dx < \infty$ for some $\alpha > 0$ and

$$\|f\| = \inf \{ \alpha > 0 : \int \varphi(|f(x)|/\alpha)\, dx \le 1 \}$$

As well known, L_φ is a Banach lattice (for details see [3] .

In case $\varphi(t) = t^p$, where $1 < p < \infty$ (i.e. $L_\varphi = L_p$), M(T) is a linear sublattice of L_φ for every positive operator $T \in \mathcal{L}(L_\varphi)$ (see [2], Theorem 2) . We shall prove a converse of this result.

<u>Theorem.</u> If M(T) is a linear subspace of L_φ for every positive operator $T \in \mathcal{L}(L_\varphi)$, then $\varphi(t) = Ct^p$, where $C > 0$ and $1 < p < \infty$.

<u>Proof.</u> We may and do assume that $\varphi(1) = 1$. For every $a,b,c,d \in \mathbb{R}$ and $f \in L_\varphi$ we put

$$U_{a,b,c,d}\, f = \left(\frac{1}{a-b} \int_a^b f\, dx \right) 1_{[c,d]} ,$$

where $1_{[c,d]}$ denotes the characteristic function of the interval $[c,d]$. By Jensen's inequality, $U_{a,b,c,d} \in \mathcal{L}(L_\varphi)$.

Fix $a,b,c > 0$ and put for $\eta, \xi \ge 0$

$$\xi_1 = \| 1_{[0,b]} \| / \| 1_{[0,c]} \| ,$$

$$R_{\eta,\xi} = \eta U_{-a,0,-a,0} + \xi U_{0,b,0,c} .$$

Jensen's inequality yields $\|\xi_1 U_{0,b,0,c}\| = 1$. Obviously, $\|R_{\eta,\xi}\| \to \infty$ as $\xi \to \infty$ for fixed η and $\|R_{\eta,\xi}\| \to \infty$ as $\eta \to \infty$ for fixed ξ. We put

$$h_{\eta,\xi}(\beta) = \frac{\|R_{\eta,\xi} f_\beta\|}{\|f_\beta\|}$$

for $\eta, \xi \geq 0$, $\beta \in [0,1]$, where $f_\beta = (1-\beta)1_{[-a,0]} + \beta 1_{[0,b]}$. Note that $h_{\eta,\xi}(\beta)$ is continuous as a function of β, η, ξ and $\|R_{\eta,\xi}\| = H_{\eta,\xi}$ where

$$H_{\eta,\xi} = \sup_\beta h_{\eta,\xi}(\beta) .$$

Indeed, since, obviously, $H_{\eta,\xi} \leq \|R_{\eta,\xi}\|$, we need to show that $\|R_{\eta,\xi}\| \leq H_{\eta,\xi}$. To this end fix a nonnegative $f \in L_\varphi$ and put $A = \frac{1}{a}\int_{-a}^{0} f\,dx$ and $B = \frac{1}{b}\int_{0}^{b} f\,dx$. We may and do assume that $A+B > 0$. Putting $\beta = B/(A+B)$ we have

$$\|R_{\eta,\xi} f\| = \|(A+B) R_{\eta,\xi} f_\beta\| \leq H_{\eta,\xi} \|(A+B) f_\beta\| .$$

Hence, by Jensen's inequality, $\|R_{\eta,\xi} f\| \leq H_{\eta,\xi} \|f\|$. Clearly, $H_{\eta,\xi}$ is continuous as a function of η and ξ. For every $\eta, \xi > 0$ there exists $\beta \in [0,1]$ with $h_{\eta,\xi}(\beta) = H_{\eta,\xi}$.

Step I. Suppose that $\inf_\beta h_{1,\xi_1}(\beta) = 1$. Then $h_{1,\xi_1}(\beta) = 1$ for all $\beta \in [0,1]$.

Indeed, in view of the definition of h_{1,ξ_1}, we have

(*) $\|(1-\beta)1_{[-a,0]} + \beta \xi_1 1_{[0,c]}\| \geq \|(1-\beta)1_{[-a,0]} + \beta 1_{[0,b]}\|$

for all $\beta \in [0,1]$. Note that equality in (*) holds for $\beta = 0$ and 1. Consider now $S \in \mathcal{L}(L_\varphi)$ defined by

$$S = U_{-a,0,-a,0} + 1/\xi_1 \, U_{0,c,0,b} .$$

Observe that $\|S\| = 1$. To this end fix a nonnegative function $f \in L_\varphi$ and put $A = \frac{1}{a}\int_{-a}^{0} f\,dx$ and $B = \frac{1}{c\xi_1}\int_{0}^{c} f\,dx$. We may and do assume that $A+B > 0$. By (*) with $\beta = B/(A+B)$ and Jensen's inequality we get

$$\|Sf\| = \|A1_{[-a,0]} + B1_{[0,b]}\| \leq \|A1_{[-a,0]} + B\xi_1 1_{[0,c]}\| \leq \|f\|.$$

It follows that $1_{[-a,0]}, 1_{[0,c]} \in M(S)$. Since, by assumption $M(S)$ is linear space, in (*) equality holds for all $\beta \in [0,1]$ and we are done.

Step II. There exist $\eta, \xi > 0$ such that $h_{\eta,\xi}$ attains its supremum at least two distinct points (i.e. there exist $\beta_1 \neq \beta_2$ in $[0,1]$ with $H_{\eta,\xi} = h_{\eta,\xi}(\beta_i)$, $i=1,2$). Suppose, to get a contradiction, that for every pair η, ξ there exists a unique β such that $h_{\eta,\xi}(\beta) = H_{\eta,\xi}$. Thus we can define a function $k: \mathbb{R}_+ \times \mathbb{R} \to [0,1]$ by $h_{\eta,\xi}(k(\eta,\xi)) = H_{\eta,\xi}$.

The function k as a function of ξ for fixed η is continuous. Indeed, let $\xi_n \to \xi_0$. We put $\beta_n = k(\eta, \xi_n)$, $n \geq 0$.

Suppose that some subsequence $\{\beta_{n'}\}$ of $\{\beta_n\}$ converges to β'. We have $\|R_{\eta,\xi_{n'}}\| = \|R_{\eta,\xi_{n'}} f_{\beta_{n'}}\| / \|f_{\beta_{n'}}\|$ and $\|R_{\eta,\xi_{n'}}\|$ $\|R_{\eta,\xi_0}\|$, so $\|R_{\eta,\xi_0}\| = \|R_{\eta,\xi_0} f_{\beta'}\| / \|f_{\beta'}\|$. By uniquness of such β we get $\beta' = k(\eta,\xi_0)$. Thus, by compactness of $[0,1]$, we obtain $\beta_n \to \beta_0$. By an analogous argument, the function $k(\cdot,\xi)$ (for fixed ξ) is continuous.

We have $h_{1,\xi_1}(0) = h_{1,\xi_1}(1) = 1$, so $H_{1,\xi_1} > 1$.

Put $\beta_{max} = k(1,\xi_1)$; obviously $h_{1,\xi_1}(\beta_{max}) > 1$. By Step I $\inf_\beta h_{1,\xi_1}(\beta) < 1$. Choose $\beta_{min} \in (0,1)$ with $h_{1,\xi_1}(\beta_{min}) = \inf_\beta h_{1,\xi_1}(\beta)$. There are two possibilities: (a) $0 < \beta_{min} < \beta_{max} < 1$, or (b) $0 < \beta_{max} < \beta_{min} < 1$.

In case (a) consider $k(1,\xi)$ as a function of ξ. We have $k(1,\xi) \neq \beta_{min}$ for all $\xi \in [0,\xi_1]$, because $\|R_{1,\xi} f_{\beta_{min}}\| < \|f_{\beta_{min}}\|$ and $\|R_{1,\xi_1}\| \geq 1$. This contradies the Darboux property of the continuous function $k(1,\cdot)$ on $[0,\xi_1]$, because $k(1,\xi_1) = \beta_{max}$ and $k(1,0) = 0$. In case (b) consider $k(\eta,\xi_1)$ as a function of η. By similar arguments we obtain a contradiction, because $k(1,\xi_1) = \beta_{max}$, $k(0,\xi_1) = 1$ and $k(\eta,\xi_1) \neq \beta_{min}$ for all $\eta \in [0,1]$.

Step III. We have

$$\|(1-\beta) 1_{[-a,0]} + \beta \xi_1 1_{[0,c]}\| = \|(1-\beta) 1_{[-a,0]} + \beta 1_{[0,b]}\|$$

for all $\beta \in [0,1]$. Indeed, by Step II there exist η, ξ, β_1, β_2 such that $\|R_{\eta,\xi} f_\beta\|$ $\|R_{\eta,\xi}\| \|f_\beta\|$ for all β $[0,1]$ and equality holds for β_1, β_2. Thus $f_{\beta_1}, f_{\beta_2} \in M(R_{\eta,\xi})$. Since, by assumption, $M(R_{\eta,\xi})$ is a linear subspace, we have $\|R_{\eta,\xi} f_\beta\| = \|R$ $\| \|f_\beta\|$ for all $\beta \in [0,1]$. In particular, for $\beta = 0$ and 1 we obtain $\eta = \|R_{\eta,\xi}\|$ and $\xi = \|R_{\eta,\xi}\| \xi_1$. Therefore $R_{1,\xi_1} = R_{\eta,\xi} / \|R_{\eta,\xi}\|$ and $\|R_{1,\xi_1} f_\beta\| = \|f_\beta\|$ for all $\beta \in [0,1]$

Step IV. Put $\psi = \varphi^{-1}$. We have

$$\psi\left(\frac{1}{b}\right) \psi\left(\frac{b}{c(a+b)}\right) = \psi\left(\frac{1}{c}\right) \psi\left(\frac{1}{a+b}\right)$$

Indeed, for every $g,h \in \mathbb{R}$ with $g < h$ we note that

$$\| 1_{[g,h]} \| = 1 / \psi(1/(h-g)).$$

Moreover, we have $\xi_1 = \psi(1/c) / \psi(1/b)$. By Step III with $\beta = 1/2$, we get $\| 1_{[-a,0]} + \xi_1 1_{[0,c]} \| = 1/\psi(1/(a+b))$

It follows that

$$a \varphi\left[\psi\left(\frac{1}{a+b}\right)\right] + c \varphi\left[\frac{\psi(1/c)}{\psi(1/b)} \psi\left(\frac{1}{a+b}\right)\right] = 1$$

which yields the desired equality.

To prove the Theorem, apply Step IV first with $b=1$ and then with $c=1$. Taking into account that $\psi(1)=1$, we get

$$\psi\left(\frac{1}{c\ a+1}\right) = \psi\left(\frac{1}{c}\right)\psi\left(\frac{1}{a+1}\right), \qquad \psi\left(\frac{1}{a+b}\right) = \psi\left(\frac{1}{b}\right)\psi\left(\frac{b}{a+b}\right)$$

It follows that

$$\psi(ts) = \psi(t)\,\psi(s)$$

for all $t,s > 0$. Since ψ is, moreover, continuous, $\psi(t) = t^{1/p}$ ([1], 2.1.2). Hence $\varphi(t) = t^p$. In view of the convexity of φ, we have $p \geqslant 1$. Since, as easily seen, the assumption of the Theorem fails for L_1, we conclude that $p > 1$.

Remark 1. The proof above uses the assumption of the Theorem for a certain family of two-dimensional operators, only.

Remark 2. The Theorem remains valued if we consider L_φ on some measurable subset Ω of $\mathbb{R}$ with $m(\Omega) > 0$. Then, in the our proof, we should use instead of the intervals $[-a,0]$, $[0,b]$, $[0,c]$ subsets X,Y,Z of Ω such that $X \cap Y = \emptyset$ and $X \cap Z = \emptyset$. Consequently, $\psi(st) = \psi(s)\psi(t)$ would hold for $t > 0$, $s > 1/m(\Omega)$. It is easy to see that $\psi(t) = t^{1/p}$ for $t > 0$, too.

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