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In: Zdeněk Frolík (ed.): Proceedings of the 10th Winter School on Abstract Analysis. Circolo Matematico di Palermo, Palermo, 1982. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 2. pp. [35]--39.

Persistent URL: http://dml.cz/dmlcz/701260

A COMBINATORIAL INTERPRETATION OF HOMOTOPY GROUPS OF POLYHEDRA

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In 1978 I showed the links between the classical homotopy groups $\Pi_n(P)$ of a polyhedron P and the regular homotopy groups $Q_n(G)$ of a finite undirected graph G (see [5], [6]).

Now I and M. Burzio have tried to obtain the same results for directed graphs (see [4]).

The aim of this research is the following fact.

If P is a polyhedron, we find a suitable triangulation T of P such that $\Pi_n(P)$ is isomorphic to $Q_n(G_T)$, where G_T is the graph of the edges of T. For example, given any triangulation U of P, we can choose T=U', where U' is the first derivated of U.

Unfortunately minimal triangulations (that is the ones with the least number of vertices) are sometimes not useful when we consider undirected graphs, while this generally does not happen for directed graphs.

For example, if we consider the circle S^1, we have:
$\Pi_n(S^1) \simeq Q_n(G_4)$, where G_4 is the graph of the edges of a square;
$\Pi_n(S^1) \not\simeq Q_n(G_3)$, where G_3 is the graph of the edges of a triangle, because all the groups $Q_n(G_3)$ are null.

Instead, if we consider the directed graph G_3' with vertices a, b, c and edges $a\to b$, $b\to c$, $c\to a$, we have $\Pi_n(S^1) \simeq Q_n(G_3')$.

Similarly, if we consider the sphere S^2, we have:
$\Pi_n(S^2) \not\simeq Q_n(G_4)$, where G_4 is the graph of the edges of a tetrahedron;
$\Pi_n(S^2) \simeq Q_n(G_6)$, where G_6 is the graph of the edges of an octahedron.

Instead we have $\Pi_n(S^2) \simeq Q_n(G_4')$, where G_4' is the directed graph with vertices a, b, c, d and edges $a\to b$, $b\to c$, $c\to d$, $d\to a$, $b\to d$, $d\to b$, $a\to c$, $c\to a$.

Now we should like to describe in a few words the construction of regular homotopy groups of finite graphs.

Definition 1. - Let G be an undirected graph. A function $f: S \to G$ is called *regular* if the inverse images $f^{-1}(v)$ and $f^{-1}(w)$ of any two different non adjacent vertices v and w of G are separated subsets of S, i.e. $\overline{f^{-1}(v)} \cap f^{-1}(w) = \emptyset = f^{-1}(v) \cap \overline{f^{-1}(w)}$.

Definition 2. - Let G be a directed graph. We call a function $f: S \to G$ *o-regular* if, for any two different vertices v and w such that $v \not\to w$ (i.e. there is no arc from v to w), we have $f^{-1}(v) \cap \overline{f^{-1}(w)} = \emptyset$. Instead we call a function $f: S \to G$ *o^*-regular* if, for any two different vertices v and w such that $v \not\to w$, we have $\overline{f^{-1}(v)} \cap f^{-1}(w) = \emptyset$.

Remark 1. The definitions of o-regular and o^*-regular function are dual, because they are interchanged by replacing G with the dually directed graph G^*.

Now let us consider an undirected graph G. If we choose as topological space S the closed interval $I = [0,1]$ of the real line $\mathbb{R}$ or a unit cube I^n, we obtain regular paths and regular loops in the graph G.

So, for any positive integer n and for any vertex v of G, we can construct the n-dimensional regular homotopy group $Q_n(G,v)$ of G on v, in the same way as in the classical case (we have only to replace the word "continuous" by the word "regular").

Moreover, if the graph G is connected, the group $Q_n(G,v)$ does not depend on the base vertex v, and we write $Q_n(G)$.

Similarly, when G is a directed graph, we obtain the o-regular homotopy groups and the o^*-regular homotopy groups. Such groups are isomorphic by a duality theorem (see [3]), since the unit cube I^n is a normal space. Therefore we will already denote them by $Q_n(G,v)$, or by $Q_n(G)$ if the graph G is weakly connected.

Now let us consider a directed graph G, and let us construct a polyhedron P_G such that $Q_n(G,v)$ is isomorphic to $\Pi_n(P_G,v)$ for any vertex v of G.

In order to do this, we give the following definitions.

Definition 3. - Let X be a subset of the graph G. A vertex of X is called a *head* (resp. a *tail*) of X in G if it is a predecessor (resp. a successor) of the other vertices of X. We say that X is *headed* (resp. *tailed*), if it has at least one head (resp. one tail). Then, if all non-empty subsets of X are headed (resp. tailed), we say that X is *totally headed* (resp. *totally tailed*).

Remark 2. - A subset of G is totally headed if and only if it is totally tailed.

Definition 4. - Let G be a directed graph. We call *complex of* G the simplicial complex K_G, whose simplices are given by the totally headed subsets of G. Then we denote by P_G the polyhedron $|K_G|$ of the complex K_G, and we call it *polyhedron of* G.

Remark 3. - If G is an undirected graph, the simplices of K_G are the complete subgraphs of G. Hence K_G is the maximal simplicial complex whose edges are the ones of G.

Now we can state the following:

THEOREM. - Let G be a finite directed graph. Then, for each positive integer n, we have:

$$Q_n(G,v) \simeq \Pi_n(P_G,v) \quad \text{for any vertex } v \text{ of } G.$$

In order to obtain this result, we need some normalization theorems, which are similar to simplicial approximation theorems for maps between polyhedra.

Such theorems permit us to choose a suitable loop in each homotopy class.

The first normalization theorem is useful to replace an o-regular function by another without singularities.

In fact we have:

Proposition 1. - Let S be a topological space and G a directed graph. A function $f:S\to G$ is o-regular if and only if, for any point $x\in S$, there is a neighbourhood U_x such that $f(U_x)$ is headed and $f(x)$ is a head of $f(U_x)$.

This proposition suggests us the following:

Definition 5. - A function $f:S\to G$ is called *completely o-regular* if, for any point $x\in S$, there is a neighbourhood U_x such that $f(U_x)$ is totally headed and $f(x)$ is a head of $f(U_x)$.

Now we can state the first normalization theorem:

"If S is a normal space, every o-regular function $f:S\to G$ is o-homotopic to a completely o-regular function $f':S\to G$."

The proof of this theorem (see [1], [2]) is very tedious.

Afterwards we give the following definitions.

Definition 6. - Let E be a set, G a graph and $\mathcal{P}=\{X_j\}(j\in J)$ a partition of E. A function $f:S\to G$ is called *quasiconstant* with respect to the partition $\mathcal{P}$ if, for each $j\in J$, the restriction of f

to X_j is constant.

Definition 7. - Let S be a topological space, G a graph, and $P=\{X_j\}(j\in J)$ a partition of S. A function $f:S\to G$ is called *weakly quasiconstant* with respect to P if, for each $j\in J$, the restriction of f to the interior $\mathring{X}_j$ of X_j is constant.

Then from the first normalization theorem we obtain the second normalization theorem:

"Let S be a compact metric space, G a directed graph, and $f:S\to G$ an o-regular function. We can find a positive real number r such that, for any partition P of S whose mesh is less than r, there is a function $g:S\to G$ which is weakly quasiconstant with respect to P, completely o-regular and o-homotopic to f."

If we consider a closed subspace S' of S and a subgraph G' of G, we can give a generalization of first and second normalization theorems. So, if S is the unit cube I^n and S' is the boundary $\dot{I}^n$ of I^n, we get the third normalization theorem:

"In each o-regular homotopy class of the group $Q_n(G,v)$, where v is a vertex of G, there exists a loop which is completely o-regular and quasiconstant with respect to a suitable triangulation of $\dot{I}^n$."

Finally we can show that $Q_n(G,v)$ is isomorphic to $\Pi_n(P_G,v)$.

We only give a short line of the proof.

First we remark that in each homotopy class we can choose a special representative. Precisely, in each classical homotopy class we find a loop which is simplicial with respect to a suitable triangulation of I^n. Similarly, in each o-regular homotopy class we take a loop that is completely o-regular and quasiconstant with respect to a suitable triangulation of I^n.

So we obtain:

(1). There is a homomorphism $\Phi^*:\Pi_n(P_G,v)\to Q_n(G,v)$.

In fact we can construct a canonical projection $p:P_G\to G$ such that, for each loop $f:I^n,\dot{I}^n\to P_G,v$, the function $pf:I^n,\dot{I}^n\to G,v$ is a completely o-regular loop. In such a way we define a function Φ from the set of continuous loops into the set of o-regular loops. Since Φ is compatible both with the homotopy relations and with the sum of loops, we obtain Φ^*.

(2). There is a homomorphism $\Psi^*:Q_n(G,v)\to\Pi_n(P_G,v)$.

In fact we choose in each o-regular homotopy class a loop f' which is completely o-regular and quasiconstant with respect to a triangulation T of I^n. Then, by considering the restriction of f'

to the vertex-set of the first derivated T' of T, we obtain an admissible vertex-transformation $w:T'\to K_G$. The linear estension $|w|:I^n\to P_G$ of w is a simplicial loop. In this way we can define Ψ^*.

(3). Φ^* is an isomorphism.

In fact we can show that $\Phi^*\Psi^*$ and $\Psi^*\Phi^*$ are the identities in $Q_n(G,v)$ and in $\Pi_n(P_G,v)$. For this end we choose as representatives of the homotopy classes the loops we mentioned before.

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