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SZEMERÉDI THEOREM IMPLIES FURSTENBERG THEOREM

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In 1936 P.Erdős and P.Turán proposed the following conjecture: Given a set A of integers with positive upper density, that is, satisfying

$$\limsup_n \frac{|A \cap [1,n]|}{n} > 0$$

then A contains arbitrarily long arithmetic progression. The conjecture was proved in 1974 by E.Szemerédi and now it is known as Szemerédi theorem / partial results were given by K.F.Roth for 3-sequences in 1952 and in 1969 by E.Szemerédi for 4-sequences/.

An equivalent formulation of Szemerédi theorem that we will use later is

For any $\omega = \{\omega_i\}$ doubly infinite sequence of zeros and ones satisfying

$$\limsup_n \frac{\sum_{i=1}^n \omega_i}{n} > 0$$

there is an arbitrarily long arithmetic progression of indexes i_k such that $\omega_{i_k} = 1$.

In 1977 Szemerédi theorem was proved by H.Furstenberg by means of ergodic theory. H.Furstenberg proved his ergodic theorem which implies Szemeredi theorem.

Let $(\Omega, \mathcal{A}, T, \mu)$ be a dynamical system, that is, $(\Omega, \mathcal{A}, \mu)$ be a probability space with σ-algebra $\mathcal{A}$ and probability measure μ, T is a measure preserving bijection $\Omega \to \Omega$.

Furstenberg ergodic theorem:

For all $A \in \mathcal{A}$, $\mu(A) > 0$ and any positive integer k there exists n such that

$$\mu[A \cap T^n A \cap \dots \cap T^{n(k-1)} A] > 0$$

A proof of the fact that Furstenberg theorem implies Szemerédi theorem may be found e.g. in [1]. We are going to prove that assuming Birkhoff ergodic theorem the theorems of Szemerédi and Furstenberg are equivalent.

1. Szemerédi theorem $\Longrightarrow$ Furstenberg theorem

At the first we formulate Furstenberg theorem in another way.

Let $(\Omega, \mathcal{A}, T, \mu)$ be given as in Furstenberg theorem.

Let $\Omega_0 = \{q\}^I$ be the space of doubly infinite sequences where

I is the set of integers.

Ω_0 equipped with a product topology is a compact and metrizable space. The subbase of the topology is formed by sets $\{\omega : \omega_n = a_n\}$ $a_n = 0$ or 1, $n \in I$ /elementary cylinders/. Let $\mathcal{A}_0$ be the least σ-algebra containing all elementary cylinders.

Let S be a shift $\Omega_0 \to \Omega_0$ / i.e. $(S\omega)_{i+1} = \omega_i$ /.

$(\Omega_0, \mathcal{A}_0, S, \mu_0)$ is a dynamical system for any probability measure μ_0 on $(\Omega_0, \mathcal{A}_0)$ preserved by S.

S is preserving μ_0 iff S is preserving μ_0 on the set of all finite intersections of elementary cylinders.

Let $\mathcal{A}'$ be the least σ-algebra containing $T^n(A), T^n(\Omega \setminus A)$ for all n from I.

$(\Omega, \mathcal{A}', T, \mu)$ is a dynamical system. It is easy to see that in order to prove Furstenberg theorem / for fixed A / one can consider only the system $(\Omega, \mathcal{A}', T, \mu)$.

Let $\psi : \Omega \to \Omega_0$ be defined by formula

$$(\psi\omega)_i = \chi_A(T^{-i}\omega).$$

Define μ_0 by $\mu_0(B) = \mu(\psi^{-1}(B))$ for $B \in \mathcal{A}_0$. It is easily seen that

1. $C \in \mathcal{A}'$ iff $\psi(C) \in \mathcal{A}_0$
2. $\psi(T\omega) = S(\psi(\omega))$

$[A \cap T^n A \cap \dots \cap T^{n(k-1)} A]$ corresponds under ψ to $\{\omega \in \Omega_0 : \omega_0 = \omega_n = \dots = \omega_{n(k-1)} = 1\}$.

Under these facts it is sufficient to prove Furstenberg theorem for $(\Omega_0, \mathcal{A}_0, S, \mu_0)$ $A = \{\omega : \omega_0 = 1\}$, $\mu_0(A) > 0$.

Suppose for some k Furstenberg theorem doesn't hold.

It is $\mu_0\{\omega : \omega_0 = \dots = \omega_{n(k-1)} = 1\} = 0$ and because of shift-invariantness of μ_0 $\mu_0\{\omega : \omega_j = \omega_{n+j} = \dots = \omega_{n(k-1)+j} = 1\} = 0$ for any j from I.

We can see that the set of all $\omega \in \Omega$ with k-arithmetic progressions of ones is a countable union of sets of measure 0 so it has measure 0. Complement of this set Ω_0' is shift invariant and of measure $\mu_0(\Omega_0') = 1$ so we can consider a dynamical system $(\Omega_0', \mathcal{A}_0', S, \mu_0)$.

Birkhoff ergodic theorem shows that for any integrable function f there exists $\hat{f}$ such that $\frac{1}{n}\sum_{i=0}^{n-1} f(S^i\omega) \to \hat{f}(\omega)$ a.s. $[\mu_0]$ and $\int f(\omega)\,d\mu_0 = \int \hat{f}(\omega)\,d\mu_0$.

Particularly consider f: $f(\omega) = \omega_0 = \chi_A(\omega)$.

We have $0 < \mu_0(A) = \int f(\omega)\,d\mu_0(\omega) = \int \hat{f}(\omega)\,d\mu_0(\omega)$.

Consequently there exists an $\omega \in \Omega_0'$ such that

$$\frac{1}{n}\sum_{i=0}^{n-1} f(S^i\omega) \longrightarrow \alpha > 0.$$

Thus there exists k-arithmetic progression ω_{γ} of ones in ω. This contradicts the assumption $\omega \in \Omega_0'$.

2. Furstenberg theorem $\Longrightarrow$ Szemerédi theorem

We have $\omega = \{\omega_n\}_{n=1}^{\infty}$ a progression of zeros and ones such that $\limsup_n \frac{1}{n}\sum_{i=0}^{n-1} \omega_i > 0$. Let $\mathcal{C}$ be an algebra of finite sums of finite intersections of elementary cylinders in Ω_0. $\mathcal{C}$ is countable and, consequently, one can choose a subsequence $\omega' = \{\omega_{n_1}, \omega_{n_1+1}, \dots, \omega_{\bar{n}_1}, \omega_{n_2}, \omega_{n_2+1}, \dots, \omega_{\bar{n}_2}, \dots\}$ $\bar{n}_i - n_i \uparrow \infty$ such that $\lim_n \frac{1}{n}\sum_{i=0}^{n-1} \chi_E(S^i\omega)$ exists for any $E \in \mathcal{C}$ and moreover $\lim_n \frac{1}{n}\sum_{i=0}^{n-1} \chi_{E_0}(S^i\omega) > 0$ where $E_0 = \{\omega : \omega_0 = 1\}$.

Put $\nu(E) = \lim_n \frac{1}{n}\sum_{i=0}^{n-1} \chi_E(S^i\omega)$ for $E \in \mathcal{C}$. It is easily seen that ν is a nonnegative and finitely additive set function on $\mathcal{C}$. Because of compactness of Ω_0 ν can be extended to a measure ν_0 defined on the whole of $\mathcal{A}_0$. $(\Omega_0, \mathcal{A}_0, S, \nu_0)$ is a dynamical system and by Furstenberg theorem for $E_0 = \{\omega : \omega_0 = 1\}$ k arbitrary integer there exists n such that

$$\nu_0[E_0 \cap S^n E_0 \cap \dots \cap S^{n(k-1)} E_0] > 0.$$

Thus

$$\frac{1}{n}\sum_{i=1}^{n} \chi_{E_0 \cap S^n E_0 \cap \dots \cap S^{n(k-1)} E_0}(S^i\omega') \longrightarrow \alpha > 0.$$

This proves Szemerédi theorem for arithmetic progressions of lenght at most $\frac{k}{2}$, k arbitrary positive integer.

Literature

Graham, R.; Rothschild, B; Spencer, J: Ramsey theory, Wiley and Sons New York 1980