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On infinite-dimensional manifolds

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Let I denote the segment $[-1,1]$, Q the Hilbert cube I^{∞} and ℓ_2 the separable Hilbert space of square-summable sequences. In [4] and [5] , the author has proved the following characterizations of spaces locally homeomorphic to Q or l_2, respectively:

1. A locally compact ANR-space, X, is a Q-manifold iff the following condition is satisfied for each n:

$(*)_n$ any map $f: I^n \times \{1,2\} \to X$ can arbitrarily closely be approximated by maps g with $g(I^n \times 1) \cap g(I^n \times 2) = \emptyset$.

2. A separable complete-metrizable ANR-space, X, is an l_2-manifold iff

$(**)$ given an open covering $\mathcal{U}$ of X and a map $f : D \to X$, where D is the disjoint union $I^0 \cup I^1 \cup \ldots$, there is a map $g: D \to X$ such that $\{g(I^n)\}_n$ is a discrete collection in X and g is $\mathcal{U}$-close to f (i.e $\{f(d), g(d)\}$ refines $\mathcal{U}$, $\forall d \in D$).

Here, we present an application of 2 to showing that certain topological groups are actually l_2-manifolds, and we report on some recent results of R.J.Daverman and J.Walsh related to result 1.

§ 1. Topological groups which are Hilbert manifolds.

T.Dobrowolski and the autor have jointly proved the following result:

Theorem ([3]). Let G be a metrizable topological group and X its separable complete-metrizable subspace which is multiplicative (i.e. $1 \in X$ and $xy \in X$ for $x,y \in X$). In order that X be an l_2-manifold it suffices that $X \in$ ANR and no neighbourhood of 1 in X be totally baunded in the right structure of G.

Combined with earlier known facts this shows the following:

Corollary 1. Let X be a complete-metrizable separable ANR. If X admits a topological group structure then either this is a Lie group structure, and X is a finite-dimensional manifold, or X is an l_2-manifold.

Corollary 2. Let X be a separable closed convex subset of a Banach space (or of a B_0-space). Then, X is either homeomorphic to l_2 or is locally compact (and then homeomorphic to one of the sets $I^k \times R^l \times [0,1)^m$ where $k \leq \infty$, $l < \infty, m \leq 1$ and $\min(m,l)=0$; see [1]).

Question: Do the analogues of the above corollaries hold true for non-separable spaces X? (C.f. the characterization of non-separable Hilbert manifolds in [5]).

Outline of the proof of the Theorem. Let d be a right-invariant metric for G. We fix $\mathcal{U}$ and $f : D \to X$ in (**) and let

$$\alpha(x) = \sup\{\mathrm{dist}_d(x, X \setminus U) : U \in \mathcal{U}\}/2, \quad x \in X,$$

$$D_k = \{d \in D : \alpha f(d) \geq 1/k\}, \quad k = 1,2,\ldots$$

Using the fact that no neighbourhood of 1 in X is totally bounded in the metric d we construct sequences $\{g_k : D \to X\}_{k \geq 0}$ and $\{\varepsilon_k\}_{k \geq 0} \subset (0,\infty)$ so that the following conditions hold for $k \geq 1$

$(1)_k$ $\quad g_k = f$ on $D \setminus D_{k+1}$ and $g_k = g_{k-1}$ on D_{k-1} ;

$(2)_k$ $\quad d(g_k(I^n \cap D_k), g_k(I^m)) > \varepsilon_k$ for $m < n$;

$(3)_k$ $\quad d(g_k, g_{k-1}) < \varepsilon_{k-1}/4$

$(4)_k \quad \varepsilon_k < \min\{1/k, \ \varepsilon_{k-1}/4\}$ (Here, $\varepsilon_0 = 1$).

It is not difficult to see that $g = \lim g_k$ is the map desired in (**). (Hint: First check that $d(g(x),f(x)) < \alpha f(x)$ for $x \in D$. To show that $\{g(I^n)\}_{n \geq 0}$ is discrete assume that $(g(x_i))_i$ converges to a point $y \in X$ and distinct x_i's belong to distinct cells in D. Then $\inf \alpha f(x_i) > 0$ – for otherwise $(f(x_i))$ would contain a sub-sequence converging to y, yielding $\alpha(y) = 0$. Thus there is a $k \in N$ with $\{x_i\}_{i=0}^{\infty} \subset D_k$ and (1) and $(2)_k$ yield $d(g(x_i),g(x_j)) \geq \varepsilon_k$ for $i \neq j$. This contradicts the assumed convergence of $(g(x_i))$.

The construction of the g_k's and ε_k's (outline). Assume for simplicity that D_1=D. In this case the sequences will terminate after the first step, which is as follows, Let ε_1 be so small that no compact set in G is an ε_1 - net in $B = \{x \in X : d(x,1) < 1\}$. We let $g_1 | I^0 = f | I^0$ and, if $g_1 | I^0 \cup \ldots \cup I^n$ is already defined, we select $p \in B$ with

$$d(p,x) > \varepsilon_1 \text{ for } x \in \{ab^{-1}: a,b, \in g_1(I^0 \cup \ldots \cup I^n) \cup f(I^{n+1})\}$$

and we put $g_1(x) = pf(x)$ for $x \in I^{n+1}$. In this way we inductively define $g_1 | I^n$ so that the resulting map g_1 satisfies $(2)_1$.

The general case (where no D_i equals D) is technically more involved but follows the same idea. See [3] for details.

§ 2. Homology characterizations of Q-manifolds. R.J.Daverman and J.Walsh have recently showed that, in the result 1, the "disjoint n-cube property" of $(*)_n$ can for $n > 2$ be replaced by a disjointness property in homologies. To be more specific let us say that, whenever (U, V) is an open pair in X and $\alpha \in H_*(U,V)$ (integer coefficients), a compact pair $(A,B) \subset (U,V)$ is said to be a Čech carier for α iff any neighbourhood of (A,B) in (U,V) contains a cycle homologous to α in (U,V).

Theorem ([2]). Let X be a locally compact ANR. Then, X is a Q-manifold iff it satisfies $(*)_2$ and the following condition:

$(*)'$ given open pairs (U_i,V_i) in X and relative cycles $\alpha_i \in H_*(U_i,V_i)$, $i = 1,2$, there are Čech carriers (A_i,B_i) for α_i with $A_1 \cap A_2 = \emptyset$.

It is unknown whether $(*)'$ is satisfied by any infinite-dimensional homology manifold X (i.e., by any locally compact ANR such that $H_*(X, X \setminus \{x\}) = 0$ for each $x \in X$). It is easy to show that $(*)'$ is satisfied if the homology manifold has the property that any relative cycle in it admits a finite-dimensional Čech carriev;see [2] . Also, property $(*)'$ is relatively easy to prove for certain CE-images of Q-manifolds, and thereby can be used to prove that these images are manifolds themselfes. A sample application is:

Corollary ([2]). If X is a space such that $X \times I^n \cong Q$ for some finite n then $X \times I^2 \cong Q$.

It is unknown whether, in the statement above, I^2 can be replaced by I. This is related to the open problem whether $X \times I$ satisfies $(*)_2$ for any infinite-dimensional homology manifold ; this problem is of great interest also for homology manifolds of finite dimension $n \geq 4$. The author has recently observed that, at least, $X \times D$ satisfies $(*)_2$ for any X as above and D a dendron with a dense set of separating points. Denoting by $p : D \to I$ the natural retraction having $\aleph_0$ non-trivial point inverses, all of which are dendra, one thus faces the following

Question: If $X \times D \cong Q$, is $1_X \times p : X \times D \to X \times I$ approximable by homeomorphisms?

References

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Editorial note:
This is an abstract of a talk presented by the author at the 8th Winter School (1980)