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Ideals in algebras of unbounded operators

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Algebras of unbounded operators appear as models in the quantum
field mechanics. One of the questions of this theory is the
description of the normal states (positive linear functionals)
of these algebras. To answer this problem W. TIMMERMANN /5/ has
introduced a class of interesting ideals. The operators of these
ideals are needed to represent the normal states as trace
functionals.

Using some latest results of the theory of locally convex spaces
and operator ideals in the main part this lecture we will give
a complete description of the geometrical properties of these
ideals. In our final remarks we will return to the problem of the
representation of linear functionals by the trace. Let us start
with some basic notations.

1. The maximal $Op^{\ast}$ – algebra $L^{+}(D)$

To exclude the very pathological cases of operator algebras the
considered operators should not be to much unbounded. In /3/
LASSNER has introduced the notion of the maximal $Op^{\ast}$-algebra on
a given domain D, which is very convinient from several reasons
and which is a straight-forward generalization of the $C^{\ast}$-algebra
$L(H)$ of the bounded operators on a Hilbert space H. Let us recall
this definition. Let D be a linear dense submanifold in a Hilbert
space H. The maximal $Op^{\ast}$-algebra $L^{+}(D)$ on the domain D is the set
of all linear operators X (bounded or not) acting in H and having
an adjoint $X^{\ast}$ such that $X(D) \subseteq D$ and $X^{\ast}(D) \subseteq D$. Of course, $L^{+}(D)$
is a $\ast$ – algebra and all operators X $L^{+}(D)$ are closable and w-
continuous in H. Important examples of domains are obtained in the
following way. Let X be a positive operator defined in H. Then
the set

$$D = \bigcap_{n=0}^{\infty} D(X^{n})$$

is the natural domain of the algebra generated by X. The most
famous example of this type is given by the operator $X = x^{2} - \dfrac{d^{2}}{dx^{2}}$,

which is generated by the position and the momentum operators. It
is well known and easy to see, that in this case the domain D
coincide with the nuclear space $\mathcal{Y}(\mathcal{R})$ of the strongly decreasing
functions on the real line. Let us return to the general case. The
algebra $L^+(D)$ defines in a natural way a topology t on D, which
is given by the set of all seminorms

$$p_X(d) := \| Xd \| \quad , \quad d \in D, \quad X \in L^+(D).$$

The domain D is called selfadjoint, if

$$D = \bigcap \{ D(X^\pi) : X \in L^+(D) \} .$$

Under this assumption (D,t) is a complete semireflexive space and
its projective spectrum can be identified with the energetic
spaces H_X of the X $L^+(D)$, where the norm in H_X is given by
$\| d \|_X^2 = \| d \|^2 + \| Xd \|^2$. The imbedding $D \longrightarrow H$ is continuous. There-
fore, we have a rigged Hilbert space

$$D \longrightarrow H \longrightarrow D_b' .$$

If (D,t) is metrizable then it is a reflexive (F)-space. This is
the case in the mentioned above examples. A countable system of
seminorms generating the topology t on D is given in this case
by $p_n(d) = \| X^n d \|$ for all $n \in \mathbb{N}$.

2. The ideals $\mathcal{A}^{\mathcal{L}}$

We assume in all the following, that D is selfadjoint and metri-
zable.
A question of some importance in the theory of quantum field
statistics is the representation of states by the trace. To solve
this problem, TIMMERMANN /5/ has introduced the following ideal
in $L^+(D)$:

$\mathcal{N}^{\mathcal{L}} = \{ T \in L^+(D) : \overline{X^* TY} \text{ is nuclear on H for all } X, Y \in L^+(D) \}$.

For each $T \in \mathcal{N}^{\mathcal{L}}$ the following definition makes sense:

$$w(X) = \text{trace } \overline{XT} = \text{trace } \overline{TX} , \quad X \in L^+(D) .$$

The linear functional w is called a trace functional. It arises
the question, what functionals on $L^+(D)$ have such a trace repre-
sentation. Before discussing this question in our final remarks,
in the main part of this section we will investigate the struc-
ture of $\mathcal{N}^{\mathcal{L}}$ or, more generally, of the following ideals:

Definition. Let $\mathcal{A}$ be an operator ideal in L(H). Then
$$\mathcal{A}^{\mathcal{L}} := \{ T \in L^+(D) : \overline{X^* TY} \in \mathcal{A} \text{ for all } X, Y \in L^+(D) \} .$$

Proposition 1. $\mathcal{A}^{\mathcal{L}}$ is an π - ideal in $L^{+}(D)$.

It is easy to see, that the operators $T \in \mathcal{A}^{\mathcal{L}}$ are bounded in fact. Moreover, we have the following characterization.

Theorem 1. An operator $T \in L^{+}(D)$ belongs to $\mathcal{A}^{\mathcal{L}}$ if and only if T is the restriction of a continuous (antilinear) operator $T_0 \colon D' \longrightarrow D$ such that all products

$$H \xrightarrow{R} D' \xrightarrow{T_0} D \xrightarrow{S} H$$

belong to $\mathcal{A}$ for all continuous linear operators R, S .

Proof. We restrict us to show, how to extend the operator T. For $h \in H$, $d \in D$ and $X \in L^{+}(D)$ it holds $|(Xd,Th)| = |(T^{\pi}Xd,h)| \le$ $\le \| T^{\pi}X \| \| d \| \| h \|$. This shows $Th \in D(X^{\pi})$. Because D is assumed to be selfadjoint, it follows $T(H) \subseteq D$. Moreover, T is continuous as a mapping from H in to D. This follows from $p_Y(Th) =$ $= \| YTh \| \le \| YT \| \| h \|$ for all $Y \in L^{+}(D)$ and all $h \in H$. The same is true for $T^{\pi} \colon H \longrightarrow D$. The dual operator $T^{\pi'}$ maps D'_b into $H' \cong H$. We show $T^{\pi'}(D') \subseteq D$. Let $d \in D$, $d' \in D'$ and $X \in L^{+}(D)$. It follows $|(Xd,T^{\pi'}d')| = |\langle T^{\pi}Xd,d' \rangle| \le c \cdot p_Y(T^{\pi}Xd) = c \| YT^{\pi}Xd \| \le c \| YT^{\pi}X \| \| d$ for some $Y \in L^{+}(D)$. This shows $T^{\pi'}d' \in D(X^{\pi})$. But D is selfadjoint. Therefore, we have $T^{\pi'}(D') \subseteq D$. The operator $T_0 = T^{\pi'}$ is the wanted extension of T because its restriction to D is T.

This theorem shows, that the operators $T \in \mathcal{A}^{\mathcal{L}}$ are bounded in a very strong sense, especially, they press H into D. It seems to be very difficult to answer the question, wether there are nontrivial (= non finite dimensional) operators in $\mathcal{A}^{\mathcal{L}}$ and what is the structure of the operators $T \in \mathcal{A}^{\mathcal{L}}$. To answer this question let us start with an example of such an operator. Let

$$\mathcal{B}_1 = \{ A \in L(H) : A(H) \subseteq D \}, \quad \mathcal{B}_1^* = \{ A^{\pi} \colon A \in \mathcal{B}_1 \} .$$

If $A \in \mathcal{B}_1$ and $X \in L^{+}(D)$, then the product XA is closed and defined on H, by the closed graph theorem it must be bounded. Now let $T_1 \in \mathcal{A}(H)$, $A,B \in \mathcal{B}_1$. Then the product $T = AT_1B^{\pi}$ belongs to $\mathcal{A}^{\mathcal{L}}$ because of $YTX = YA \cdot T_1 \cdot (X^{\pi}B)^{\pi} \in \mathcal{A}(H)$ for all $X,Y \in L^{+}(D)$. It is very surprising that this example covers the general case:

Theorem 2. If $(\mathcal{A}, \alpha)$ is a normed, complete ideal in $L(H)$, then

$$\mathcal{A}^{\mathcal{L}} = \mathcal{B}_1 \cdot \mathcal{A} \cdot \mathcal{B}_1^{\pi} .$$

The proof of this theorem is based on a deep result of the theory of locally convex spaces. First of all we need the following proposition.

Proposition 2. Let F be a metrizable locally convex space having a neighbourhood base $\{U_n\}$ such that all spaces F_{U_n} are Hilbert spaces of dimension $\leqq \aleph$. Then there is a fundamental system of absolutely convex bounded subsets $\{M_\alpha\}$ such that the spaces F_{M_α} are Hilbert spaces of dimension $\leqq \aleph$.

Proof. Let K be any bounded subset of F. Let p_{U_n} be the gauge functional of U_n. We put $c_n = \sup\{p_{U_n}(x); \ x \in K\}$ and define

$$M = \left\{ x \in F; \ p_M(x)^2 = \sum_{n=1}^{\infty} 2^{-n} c_n^{-1} p_{U_n}(x)^2 \leqq 1 \right\} .$$

This set is bounded and contains K. But p_M satisfies the parallelogram equation, therefore, F_M must be a Hilbert space of dimension $\leqq \aleph_0 \cdot \aleph = \aleph$.

Concerning some notions in the following deep theorem we refer to /4/. In our application to Hilbert spaces the assumed injectivity and surjectivity is not a restricting condition.

Theorem 3. (/1, thm. 7.1.8/). Let $(\mathcal{A}, \alpha)$ be a normed complete operator ideal in the class of Banach spaces which is surjective and injective. Let DF be a barrelled (complete) (DF)-space and F be an (F)-space. If T: DF $\longrightarrow$ F is a linear continuous operator such that the products

$$B_0 \xrightarrow{R} DF \xrightarrow{T} F \xrightarrow{S} B_1$$

belong to $\mathcal{A}$ for all Banach spaces B_0, B_1 and all linear continuous operators R and S, then T admits a linear continuous factorization

$$
\begin{array}{ccc}
DF & \xrightarrow{\quad T \quad} & F \\
\downarrow & & \uparrow \\
B_2 & \xrightarrow{\quad T_1 \quad} & B_3
\end{array}
$$

through Banach spaces B_2, B_3 and an operator $T_1 \in \mathcal{A}$.

Sketch of the proof of theorem 3. We extend the ideal $(\mathcal{A}, \alpha)$ in L(H) to an injective and surjective quasinormed ideal in the class of all Banach spaces. Let $T \in \mathcal{A}^{\mathcal{L}}$. Its extension T_0 according to theorem 1 satisfies the assumption of theorem 3. Therefore, T_0 has a factorization through $T_1 \in \mathcal{A}$. Using proposition 2 we can replace B_2 and B_3 by the Hilbert space H. This yields a factorization

$$
\begin{array}{ccc}
D' & \xrightarrow{T_0} & D \\
Q' \downarrow & & \uparrow P \\
H & \xrightarrow{T_1} & H
\end{array}
$$

with $Q, P \in \mathcal{B}_1$. Restricting T_0 to D we obtain $T_0 \mid D = T = PT_1Q^{\textbf{x}}$.

3. Final remarks
Dual to the definition of $\mathcal{A}^{\mathcal{L}}$ you can define the set
$$\mathcal{A}^{\mathcal{B}} = \left\{ S \in L^+(D) : \mathcal{B}_1^{\textbf{x}} \cdot S \cdot \mathcal{B}_1 \subsetneqq \mathcal{A} \right\}.$$
This set is also an ж - ideal in $L^+(D)$, in generally it contains unbounded operator. There is the following theorem concerning the structure of $\mathcal{A}^{\mathcal{B}}$. An ideal $\mathcal{A}$ is called perfect, if it coincides with its second adjoint ($\mathcal{A} = \mathcal{A}^{\textbf{xx}}$, see /4/).

Theorem 4. If $(\mathcal{A}, \alpha)$ is a perfect ideal in L(H), then
$$\mathcal{A}^{\mathcal{B}} = L^+(D)' \cdot \mathcal{A} \cdot L^+(D) ,$$

where $L^+(D)' = \left\{ X' : D' \longrightarrow D' \mid X \in L^+(D) \right.$.

Now we return to the problem of the trace representation of the linear functionals on ideals.

Proposition 3. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two normed complete ideals in L(H) such that their product $\mathcal{A}_1 \cdot \mathcal{A}_2$ contains only nuclear operators. Then a dual pair $\langle \mathcal{A}_1^{\mathcal{L}}, \mathcal{A}_2^{\mathcal{B}} \rangle$ is defined by
$$\langle T, S \rangle = \langle PT_1Q^{\textbf{x}}, S \rangle := \text{trace } Q^{\textbf{x}}SP \cdot T_1 .$$

Uring this dual pair it is now possible to define dual pair topologies on $\mathcal{A}_1^{\alpha}$ and $\mathcal{A}_2^{B}$. Then the linear continuous functionals are exactly the trace functionals.

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