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PRODUCTS OF IDEALS OF BOREL SETS

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The natural definition of the product $\mathcal{I} \times \mathcal{J}$ of ideals in the fields $\mathcal{B}(X)$, $\mathcal{B}(Y)$ of all Borel sets in topological spaces X, Y sounds as follows:

for any $A \in \mathcal{B}(X \times Y)$ we set

$$A \notin \mathcal{I} \times \mathcal{J} \equiv \{x \in X; \{y \in Y; (x, y) \in A\} \notin \mathcal{J}\} \notin \mathcal{I}.$$

This definition is meaningful if

(*) the sets $\{y \in Y; (x, y) \in A\}$ for $x \in X$ are Borel in Y , and if

(**) the set $\{x \in X; \{y \in Y; (x, y) \in A\} \notin \mathcal{J}\}$ is Borel in X .

The first condition is always satisfied, the second one depends on the ideal \mathcal{J} .

Let us denote by \mathcal{L}, \mathcal{K} the ideals of all Borel sets in the real unit interval I , of the Lebesgue measure zero, or of the first Baire category, respectively.

Theorem 1. If the ideal \mathcal{J} is a product of finitely many ideals, each equal to \mathcal{L} , or to \mathcal{K} , then the condition (**) is satisfied.

Theorem 1 enables us to form products of ideals \mathcal{L}, \mathcal{K} , in arbitrary order. The following theorem describes an important property of such products.

Theorem 2. If the ideal \mathcal{J} is the product of m ideals, each equal to \mathcal{L} or to \mathcal{K} , then \mathcal{J} is countably complete and the boolean algebra $\mathcal{B}(I^m)/\mathcal{J}$ fulfills the countable chain condition. The algebra $\mathcal{B}(I^m)/\mathcal{J}$ is, therefore, complete.

Complete boolean algebras and their complete boolean products are closely connected with boolean-valued models of the axiomatic set theory. In [1], [2] the property of local disjointness is described, which is fulfilled in a complete boolean product if and only if the corresponding model classes are disjoint over the basic model.

Theorem 3. If the ideal \mathcal{I} is the product of m ideals, k of which (not necessarily the first ones) are equal to \mathbb{L} and $m-k$ are equal to \mathbb{K} , $0 < k < m$, then the complete product $\mathcal{B}(\mathbb{I}^m)/\mathcal{I}$ of algebras $\mathcal{B}(\mathbb{I}^k)/\mathbb{L}^k$, $\mathcal{B}(\mathbb{I}^{m-k})/\mathbb{K}^{m-k}$ induced by the natural embeddings, is locally disjoint.

Remarks. 1. By the well-known Fubini's theorem, the algebra $\mathcal{B}(\mathbb{I}^k)/\mathbb{L}^k$ is isomorphic to the so-called random algebra $\mathcal{R} = \mathcal{B}(\mathbb{I})/\mathbb{L}$. Analogously, $\mathcal{B}(\mathbb{I}^{m-k})/\mathbb{K}^{m-k}$ is isomorphic to the Cantor algebra $\mathcal{C} = \mathcal{B}(\mathbb{I})/\mathbb{K}$. Thus, Theorem 3 is a tool for constructing infinitely many non-isomorphic locally disjoint products of algebras \mathcal{R}, \mathcal{C} .

2. The product of algebras \mathcal{R}, \mathcal{C} , described above are non-isomorphic when considered as products. It is a problem, if they are isomorphic as boolean algebras. E.g. are the boolean algebras $\mathcal{B}(\mathbb{I}^2)/\mathbb{L} \times \mathbb{K}$, $\mathcal{B}(\mathbb{I}^2)/\mathbb{K} \times \mathbb{L}$ isomorphic?

References

- [1] L. Bukovský, Cogeneric extensions, Proc. Wrocław Logic Coll. 1977, North Holland 1978, 91-98.
- [2] M. Gavalec, Local properties of complete boolean products, to appear in Coll. Math.