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PRODUCTS OF IDEALS OF BOREL SETS

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The natural definition of the product $\mathcal{I} \times \mathcal{J}$ of ideals in the fields $\mathcal{B}(X)$, $\mathcal{B}(Y)$ of all Borel sets in topological spaces X, Y sounds as follows:

for any $A \in \mathcal{B}(X \times Y)$ we set

$$A \in \mathcal{I} \times \mathcal{J} \equiv \{x \in X ;\ \{y \in Y ;\ (x,y) \in A\} \notin \mathcal{J}\} \in \mathcal{I}.$$

This definition is meaningful if

(*) the sets $\{y \in Y ;\ (x,y) \in A\}$ for $x \in X$ are Borel in Y, and if

(**) the set $\{x \in X ;\ \{y \in Y ;\ (x,y) \in A\} \notin \mathcal{J}\}$ is Borel in X.

The first condition is always satisfied, the second one depends on the ideal $\mathcal{J}$.

Let us denote by $\mathbb{L}, \mathbb{K}$ the ideals of all Borel sets in the real unit interval I, of the Lebesgue measure zero, or of the first Baire category, respectively.

Theorem 1. If the ideal $\mathcal{J}$ is a product of finitely many ideals, each equal to $\mathbb{L}$, or to $\mathbb{K}$, then the condition (**) is satisfied.

Theorem 1 enables us to form products of ideals $\mathbb{L}, \mathbb{K}$, in arbitrary order. The following theorem describes an important property of such products.

Theorem 2. If the ideal $\mathcal{J}$ is the product of n ideals, each equal to $\mathbb{L}$ or to $\mathbb{K}$, then $\mathcal{J}$ is countably complete and the boolean algebra $\mathcal{B}(I^n)/\mathcal{J}$ fulfills the countable chain condition. The algebra $\mathcal{B}(I^n)/\mathcal{J}$ is, therefore, complete.

Complete boolean algebras and their complete boolean products are closely connected with boolean-valued models of the axiomatic set theory. In [1], [2] the property of local disjointness is described, which is fulfilled in a complete boolean product if and only if the corresponding model classes are disjoint over the basic model.

<u>Theorem 3</u>. If the ideal $\mathcal{J}$ is the product of n ideals, k of which (not necessarily the first ones) are equal to $\mathbb{L}$ and $n-k$ are equal to $\mathbb{K}$, $0 < k < n$, then the complete product $\mathcal{B}(I^n)/\mathcal{J}$ of algebras $\mathcal{B}(I^k)/\mathbb{L}^k$, $\mathcal{B}(I^{n-k})/\mathbb{K}^{n-k}$ induced by the natural embeddings, is locally disjoint.

<u>Remarks</u>. 1. By the well-known Fubini's theorem, the algebra $\mathcal{B}(I^k)/\mathbb{L}^k$ is isomorphic to the so-called random algebra $\mathcal{R} = \mathcal{B}(I)/\mathbb{L}$. Analogously, $\mathcal{B}(I^{n-k})/\mathbb{K}^{n-k}$ is isomorphic to the Cantor algebra $\mathcal{C} = \mathcal{B}(I)/\mathbb{K}$. Thus, Theorem 3 is a tool for constructing infinitely many non-isomorphic locally disjoint products of algebras $\mathcal{R}$, $\mathcal{C}$.

2. The product of algebras $\mathcal{R}$, $\mathcal{C}$, described above are non-isomorphic when considered as products. It is a problem, if they are isomorphic as boolean algebras. E.g. are the boolean algebras $\mathcal{B}(I^2)/\mathbb{L}\times\mathbb{K}$, $\mathcal{B}(I^2)/\mathbb{K}\times\mathbb{L}$ isomorphic ?

References

[1] L.Bukovský, Cogeneric extensions, Proc. Wroclaw Logic Coll. 1977, North Holland 1978, 91-98.

[2] M.Gavalec, Local properties of complete boolean products, to appear in Coll. Math.