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Let us recall that a Banach space X has the uniform bounded approximation property if and only if there exists a function $k : \mathbb{N} \to \mathbb{N}$ and a constant $C > 0$ such that for every finite-dimensional subspace $F \subset X$ there exists a subspace $E \subset X$ and a linear operator $S : X \to E$ such that $E \supset F$, $S|F = Id_F$ and $\|S\| \leq C$, $\dim(E) \leq k(\dim(F))$.

Bożejko and Pełczyński [1] showed that the space $L^1(G)$, where G is a compact abelian group, has an analogue of the uniform bounded approximation property. Exactly, they proved this property for the translation invariant operators and translation invariant subspaces.

Now, we shall prove a similar property for the Haar basis in the space L^1. Let us denote by $\mathcal{W}$ the set of elements of the Haar basis and for $A \subset \mathcal{W}$ let us denote

$$L^1_A = \left\{ f \in L^1 : f = \sum_{\chi \in \mathcal{W}} a_\chi \cdot \chi \ \text{ and } \ a_\chi = 0 \ \text{ for } \ \chi \notin A \right\}.$$

For $\chi \in \mathcal{W}$ let $I_\chi = \{x \in I : \chi(x) \neq 0\}$, where $I = [0,1]$. We shall say that for $\chi_1, \chi_2 \in \mathcal{W}$ element χ_1 is less than element χ_2 $(\chi_1 < \chi_2)$ iff $I_{\chi_1} \subseteq I_{\chi_2}$.

<u>Theorem.</u> For every finite subset $A \subset \mathcal{W}$ there exists a subset $B \subset \mathcal{W}$ and a linear operator $S : L^1_A \to L^1_B$ such that

$$B \supset A, \ |B| \leq 9 \cdot |A|, \ S|L^1_A = Id_{L^1_A} \ \text{ and } \ \|S\| \leq 5 .$$

Proof. We shall construct the set B at first. Let us add to the set A all elements χ of the set $\mathcal{W}$ such that the both branches of elements less than χ contain elements of the set A . We obtain some set $B^* = H(B)$. We shall prove, by the induction method, that $|B^*| \leq 3.|A|$.

Actually, let the element χ_o be minimal in the set B . We apply the inductive hypothesis to the set $B_o = B \setminus \{\chi_o\}$. We have $|B_o^*| = |H(B_o)| \leq 3.|B_o|$. Moreover $H(B) \subset H(B_o) \cup \cup \{\chi_o, \chi_1, \chi_2\}$ where χ_1 and χ_2 are the elements following in the order relation the element χ_o . So

$$|B^*| \leq |B_o^*| + 3 \leq 3.|B_o| + 3 = 3.|B| .$$

Let $B^* = \bigcup_{j=1}^{m} B_j^*$ be a partition of the set B onto its components (maximal connected subsets in the order relation). Let H_j $(j=1,2,\dots,m)$ be the set of all elements of $\mathcal{W}$, which are greater than all elements of B_j . Let us take the set G_j of minimal elements of H_j and define the set $B = B^* \cup \bigcup_{j=1}^{m} G_j$. From the construction it follows that $|B| \leq 3.|B^*|$.

The set B has the following properties:

1. $B \supset A$, $|B| \leq 9.|A|$.
2. $B = \bigcup_{j=1}^{m} B_j$ where the sets B_j are disjoint and connected.
3. For every element B_j $j=1,2,\dots,m$ both elements χ_1 and χ_2 following χ in the set $\mathcal{W}$ belong to B_j or both these elements don't belong to B_j .
4. For every maximal element χ in B_j only one of two

branches of elements less than χ may contain elements of the set B.

Now we may define an operator S. Let $S(\chi_0) = 0$ for $\chi_0 \notin B$, $S(\chi_0) = \chi_0$ for $\chi_0 \in B_j$ not maximal in B_j. For $\chi_0 \in B_j$ maximal in B_j let us consider two branches of the elements of $\mathcal{W}$ less than χ_0. Let us take this branch, which contains elements of the set B. If such branch does not exist, we choose an arbitrary branch.

Let us take an arbitrary element χ from the chosen branch and $a = \int_I \chi_0 . e_\chi dt$, where e_χ is the characteristic function of the interval I_χ. We define $S(\chi_0) = -\frac{F(e_\chi)}{a}$, where F is a projection $F(\sum_{\chi \in \mathcal{W}} a_\chi . \chi) = \sum_{\chi \in B_j \setminus \{\chi_0\}} a_\chi . \chi$.

This definition does not depend on the element χ. Since the set B_j is connected, we have $\|F\| \leq 2$.

We shall prove that $\|S\| \leq 4$. For this purpose it is sufficient to show that $\|S(e_\chi)\| \leq 4.\|e_\chi\|$ for $\chi \in \mathcal{W}$.

Let us denote $P = \{\chi^* \in B : \int_I \chi^* e_\chi dt \neq 0\}$. We have $S(e_\chi) = \sum_{j=1}^{m} S(F_j(e_\chi))$ where F_j (for $j=1,2,\dots,m$) are projections $F_j(\sum_{\chi \in \mathcal{W}} a_\chi . \chi) = \sum_{\chi \in B_j} a_\chi . \chi$. Moreover, by the definition, $S(F_j(e_\chi)) = 0$ if there is an element χ in P greater then all elements of $B_j \cap P$. But the set P is a chain, so only for one number j it may happen that $S(F_j(e_\chi)) \neq 0$. So let $S(e_\chi) = S(F_{j_0}(e_\chi))$ and let χ_0 be maximal element of $P \cap B_{j_0}$ and $P : L^1 \to L^1$ be a projection

$P(\sum_{\chi \in w} a_\chi \cdot \chi) = \sum_{\chi \in B_{j_o} \setminus \{\chi_o\}} a_\chi \cdot \chi$. We have $S(F_{j_o}(e_\chi)) =$

$= a.S(\chi_o) + S(P(e_\chi)) = a.S(\chi_o) + P(e_\chi)$ where $a =$

$= \int_I \chi_o \cdot e_\chi dt$. So

$$\|S(e_\chi)\| = \|S(F_{j_o}(e_\chi))\| \leq a.\|S(\chi_o)\| + \|P(e_\chi)\| \leq 4.\|\chi_o\|$$

since $\|P\| \leq 2$, $a.\|\chi_o\| \leq \|e_\chi\|$ and $\|S(\chi_o)\| \leq 2.\|\chi_o\|$.

References

[1] Bożejko M. and Pełczyński A.: An Analogue in Commutative Harmonic Analysis of the Uniform Bounded Approximation Property of Banach Space. Seminaire D'Analyse Fonctionnelle 1978-1979, Expose No. IX