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Does 2^X exist for a proper class X?

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To explain the title we have to specify what we mean under 2^X. We will work in the Gödel-Bernays set theory with the axiom of choice for sets and without the axiom of regularity. We will denote it by GB_0; GB will be GB_0 with the axiom of regularity added. Concerning the terminology, having a relation R and a set x then the class $\{y / \langle x,y\rangle \in R\}$ will be denoted by $Ext_R(x)$. The class $D(R) = \{x / Ext_R(x) \neq \emptyset\}$ is the domain of R. A relation R is called <u>nowhere constant</u> if $Ext_R(x) \neq Ext_R(y)$ for any $x,y \in D(R)$, $x \neq y$.

It is easy to verify that the following two conditions are equivalent for any class A:

(1) There is a class 2^A and a mapping $E_A: 2^A \times A \longrightarrow 2$ such that for any class C and any mapping $F: C \times A \longrightarrow 2$ there is a unique mapping $\bar{F}: C \longrightarrow 2^A$ such that $E_A \circ (\bar{F} \times 1) = F$.

(2) There exists a nowhere constant relation E_A such that for any non-empty subclass $\emptyset \neq X \subseteq A$ there is $x \in D(E_A)$ with $X = Ext_{E_A}(x)$.

<u>Definition:</u> A class A will be called <u>small</u> if the equivalent conditions (1) and (2) are satisfied.

If A is a set then condition (1) is the description of the set of subsets of A in category theory terms. Condition (2) says that E_A codes subclasses of A and precisely this coding is used e.g. in [5].

<u>Proposition 1:</u> (i) Any set is a small class.

(ii) If $H: A \longrightarrow B$ is an injective mapping and B a small class then A is small.

(iii) If $H: A \longrightarrow B$ is a surjective mapping and A is small then B is small.

(iv) If n is a natural number and $A_1, \dots A_n$ are small classes then

$A_1 \cup \ldots \cup A_n$ is small.

(v) If A is a small class then there is no surjective mapping from A onto the class V of all sets.

Hence if any proper class is bijective with V then sets are the only small classes. However, the existence of small proper classes is consistent with GB_0. To show it we will follow the indication of A.Sochor and will use the permutation model (see e.g. [1]).

Example: Let us start with a model of GB_0 with a proper class A of atoms such that V = Ker(A). By a permutation of A we mean a set which is a bijective mapping of some set $X \subseteq A$ into itself. Every permutation of A extends to an $\in$-automorphism of V. A class C is symmetric if there is a finite subset $\{a_1, \ldots, a_n\} \subseteq A$ such that f(C) = C for any permutation f such that $f(a_i) = a_i$ for $i = 1, \ldots, n$. A class C is hereditarily symmetric if it is symmetric any any element of its transitive closure is symmetric. Hereditarily symmetric classes form a model M of GB_0.

It is easy to see that hereditarily symmetric subclasses of A are precisely finite subclasses of A and complements of finite subclasses of A relative to A. Hence A is small in M because

$E_A = \{\langle\langle x,0\rangle ,y\rangle \;/\; x \in P_{fin}(A),\ y \in x\} \cup \{\langle\langle x,1\rangle ,y\rangle \;/\; x \in P_{fin}(A),\ y \notin x\}$

is hereditarily symmetric. Here $P_{fin}(A)$ denotes the class of all finite subclasses of A.

Problem: Is the existence of small proper classes consistent with GB?

Small proper classes could be used in the foundation of the category theory. One needs there manipulations with proper classes which are not allowed by GB. However, the means of GB could be sufficient for categories which are small in the sense that their morphisms form a small class. To do it we would need stronger axioms of the existence of small classes. E.g.: A small $\Longrightarrow$ 2^A small. This axiom ensures that for any two small classes A,B there is a small class A^B which codes mappings $B \longrightarrow A$. Hence for small categories A,B there exists then the functor category A^B which is itself small. Moreover, if the class Card of all cardinals is small then the category of sets, which is not small, is equivalent to a small category having cardinals as objects. Then the most of the usual categories of mathematical structures would be equivalent to small categories and there could hence be treated in the realm of GB.

The similar idea was developped in GB_o by Osius [4]. He uses atoms to code certain proper classes which allows him to manipulate these C-classes in the same way as sets. His C-classes are small in our sense and stronger axioms of their existence are also satisfied. Especially, his paper provides another model showing the consistency of the existence of small proper classes with GB_o.

The equivalence of conditions (1) and (2) holds also in the theory TSS of semisets of Vopěnka and Hájek [5]. We will use their definitions and results without further explanations, we indicate only that the basic difference between GB and TSS lies in the fact that subclasses of sets (i.e. <u>semisets</u>) need not be sets. The assertion (i) of Proposition 1 need not hold now. It is easy to see that in the theory (TSS',E1,St) the supposition that E_A is nowhere constant may be omitted in (2) for any semiset A. Hence the satisfaction of (2) for any semiset A is precisely the axiom (Pot) of [5].

<u>Proposition 2</u> (TSS',E1,St,Pot): Any semiset is a small class.

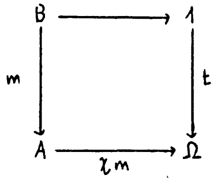
A category T is a <u>topos</u> (see e.g. [2]) if it satisfies the following conditions:

(i) T has finite limits

(ii) T has a subobject classifier Ω

(iii) A power-object Ω^A exists for each object A of T.

Here (ii) means that there is a morphism t: $1 \longrightarrow \Omega$ from the terminal object 1 of T such that for any monomorphism m: $B \longrightarrow A$ there exists a unique morphism χm (the characteristić morphism of m) such that



is a pull-back. Further, (iii) means (1) if we write Ω instead of 2, object instead of class and morphism instead of mapping.

Denote by P the category of all semisets in a given model **M** of (TSS', E1, St, Pot). Morphisms are mappings which are semisets (of course, P is not a category in the sense of M). Then P evidently has finite limits and 2 is a subobject classifier. Hence Proposition 2 yields

<u>Corollary:</u> P is a topos.

Further, since P satisfies axioms (ND), (G) and (ES) from [2], P is equivalent to the category of sets of some model of a Zermelo--Fraenkel set theory without the axiom scheme of replacement.

Let Cl be the category of all classes in M. It is easy to see that the existence of a totally universal relation (in the sense of [5]) in M is equivalent with the fact that Cl satisfies the axiom (A 19) from Osius [3] where small objects in Cl (in the sense of [3]) are semisets. Hence if M is a model of (TSS', E1, S5) then semisets and classes form a model of the Osius theory of the category of classes.

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