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The Kantorovič-Rubinstein distance

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The Kantorovič-Rubinstein distance

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If (X,ρ) is a pseudometric space then the pseudometric ρ naturally induces some distance between probability measures. There are the following natural possibilities how to define such a distance :

$$\rho_1(\mu,\nu)=\sup\,\{\mu(f)-\nu(f) \mid \mathrm{Lip}(f)\leq 1,\ f \text{ is bounded}\},$$

$$\rho_2(\mu,\nu)=\inf\,\{\eta(\rho) \mid \pi_1\eta-\pi_2\eta=\mu-\nu\ ,\ \eta \text{ is a measure on } \Sigma\otimes\Sigma\},$$

$$\rho_3(\mu,\nu)=\inf\,\{\eta(\rho) \mid \pi_1\eta=\mu,\ \pi_2\eta=\nu\ ,\ \eta \text{ is a probability measure on } \Sigma\otimes\Sigma\},$$

where μ,ν are probability measures on a σ-algebra Σ on X containing all Borel sets $(\mu(f)=\int f\,d\mu\ ,\ \pi_1\eta(A)=\eta(A\times X)$ etc.).
The last two metrics are usually called the Kantorovič-Rubinstein distances.

Under certain conditions all metrics given above are equal. That's the reason why it is convenient to work with the Kantorovič-Rubinstein distance.

Kantorovič has shown in [1] that $\rho_1=\rho_2$ if X is a compact metric space. In [2] and [3] Kantorovič and Rubinstein proved (essentially) that $\rho_1=\rho_2$ if X is a compact metric space. We shall show that $\rho_1=\rho_2=\rho_3$ if X is an arbitrary separable pseudometric space.

We shall use the following theorem on a non-negative extension of a linear functional.

<u>Theorem 1.</u> Let E be an ordered vector space, let I be a non-negative linear functional on a subspace F of E. Let

$$(\forall y\in E)(\exists z\in F)\quad y\leq z$$

(i.e. F is a majorizing subspace of E).

Then there is a non-negative linear extension of I to E.

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Theorem 2. t (,ρ b a e abl p eu o tric space, let μ, ν be two probability measures on a σ-algebra Σ on X containing all Borel sets. Let ρ_1, ρ_2, ρ_3 be the distances defined above.

Then $\rho_1 = \rho_2 = \rho_3$.

Proof: (1) Let η be a measure on $\Sigma \otimes \Sigma$ such that $\pi_1\eta - \pi_2\eta = \mu - \nu$. Then it holds

$\mu(f) - \nu(f) = \int f . d\pi_1\eta - \int f . d\pi_2\eta = \int f(x)\, d\eta(x,y) - \int f(y)\, d\eta(x,y) \leq \int \rho\, d\eta = \eta(\rho)$ for each bounded function f with $\mathrm{Lip}(f) \leq 1$.

Thus we have $\rho_1 \leq \rho_2$. Obviously $\rho_2 \leq \rho_3$.

If $\rho_1 = \infty$, then $\rho_1 = \rho_2 = \rho_3 = \infty$. Thus let us consider the case $\rho_1 < \infty$.

(2) Lemma. Let φ, ψ be bounded measurable functions, $a \in R$ and

$\varphi(x) + \psi(y) + a\rho(x,y) \geq 0$ for all $x, y \in X$.

Then it holds $\mu(\varphi) + \nu(\psi) + a\rho_1 \geq 0$.

Proof of the lemma:

a) For $a \leq 0$ we have

$\mu(\varphi) + \nu(\psi) = \int \varphi(x)\, d\eta(x,y) + \int \psi(y)\, d\eta(x,y) \geq (-a) \int \rho(x,y)\, d\eta(x,y) \geq$
$\geq -a\rho_3 \geq -a\rho_1$, where $\eta = \mu \otimes \nu$ (the last inequality is valid by virtue of (1)) .

b) Let $a > 0$. Put $h(x) = \inf \{ \psi(y) + a\rho(x,y) \mid y \in X \}$.

Then substituting y=x we get $h(x) \leq \psi(x)$ and by the assumption $h(x) \geq -\varphi(x)$, hence h is a bounded function.

For a fixed $y \in X$

$$\psi(y) + a\rho(x,y)$$

is a Lipschitz function with the constant a, thus $\mathrm{Lip}(h) \leq a$ as well. Hence we have

$$a\rho_1 \geq \mu(h) - \nu(h) \geq -\mu(\varphi) - \nu(\psi).$$

(3) By means of the lemma one can easily show that the formula

$$\tilde{\eta}(f) = \mu(\varphi) + \nu(\psi) + a\rho_1$$

gives a sound definition of a non-negative linear functional for all $f(x,y)=\varphi(x)+\psi(y)+a\rho(x,y)$, where φ,ψ are bounded measurable functions and $a\in\mathbf{R}$.

By the theorem 1 and by the lemma there is a non-negative linear extension of $\tilde{\eta}$ to all functions majorized (in the absolute value) by $a\rho(x,y)+b$ (where a, b are positive constants). We shall denote the extension by $\tilde{\eta}$ as well.

(4) For an arbitrary $\varepsilon>0$ there is a sequence of pairwise disjoint sets $A_n\in\Sigma$ such that $\operatorname{diam} A_n<\varepsilon$ and $\bigcup_{n=1}^{\infty}A_n=X$, for (X,ρ) is a separable space.

Put

$$c_{ij}=\begin{cases}0 & \text{for } \mu(A_i)\cdot\nu(A_j)=0\\ \dfrac{\tilde{\eta}(A_i\times A_j)}{\mu(A_i)\nu(A_j)} & \text{otherwise,}\end{cases}$$

and

$$\eta(B)=\sum_{i,j=1}^{\infty}c_{ij}\,\mu\otimes\nu\,(B\cap(A_i\times A_j))\ .$$

Then η is a non-negative σ-additive measure on $\Sigma\otimes\Sigma$.

Furthermore we have

$$\eta(A\times X)=\sum_{\mu(A_i)\nu(A_j)\neq 0}\frac{\tilde{\eta}(A_i\times A_j)}{\mu(A_i)\nu(A_j)}\cdot\mu\otimes\nu((A\cap A_i)\times A_j)=$$

$$=\sum_{\mu(A_i)\neq 0}\frac{\tilde{\eta}(A_i\times A_j)}{\mu(A_i)}\cdot\mu(A\cap A_i)\ .$$

for all $A\in\Sigma$ (if $\nu(A_j)=0$ then $\tilde{\eta}(A_i\times A_j)\leq\tilde{\eta}(X\times A_j)=\nu(A_j)=$ $=0$, thus $\tilde{\eta}(A_i\times A_j)=0$).

(5) Denote $B_n=\bigcup_{k=1}^{n}A_k$. Then it holds

$$0\leq\tilde{\eta}(A_i\times X)-\sum_{j=1}^{n}\tilde{\eta}(A_i\times A_j)\leq\tilde{\eta}(A_i\times(X-B_n))\leq\tilde{\eta}(X\times(X-B_n))=$$
$$=\nu(X-B_n),$$

but $(X-B_n)\searrow\emptyset$, hence

$$\sum_{j=1}^{\infty}\tilde{\eta}(A_i\times A_j)=\tilde{\eta}(A_i\times X)=\mu(A_i).$$

Thus we have

$$\eta(A\times X)=\sum^{\infty}\mu(A\cap A_i)=\mu(A)\qquad\text{for all } A\in\Sigma\ (\text{if } \mu(A_i)=0$$

then obviously $\tilde{\eta}(A_i \times A_j)=0$),

analogously $\eta(X \times A)=\nu(A)$ for all $A \in \Sigma$, i.e. $\pi_1\eta=\mu$, $\pi_2\eta=\nu$.

(6) Put $\bar{\rho} = \sup \rho | A_i \times A_j$ on $A_i \times A_j$ and analogously

$\underline{\rho} = \inf \rho | A_i \times A_j$ on $A_i \times A_j$.

Then it holds

$$\eta(\rho) \le \eta(\bar{\rho}) \le \eta(\underline{\rho})+2\varepsilon \le \tilde{\eta}(\rho)+2\varepsilon=\rho_1+2\varepsilon,$$

for $\eta(\underline{\rho}\cdot c_{B_n\times B_n}) \to \eta(\underline{\rho})$ and

$$\eta(\underline{\rho}\cdot c_{B_n\times B_n})=\sum_{i,j=1}^{n} c_{ij}\,\mu\otimes\nu(A_i\times A_j).\inf \rho|A_i\times A_j =$$

$$=\sum_{i,j=1}^{n}\tilde{\eta}(A_i\times A_j).\inf \rho|A_i\times A_j = \sum_{i,j=1}^{n}\tilde{\eta}(\underline{\rho}\cdot c_{A_i\times A_j}) =$$

$$=\tilde{\eta}(\underline{\rho}\cdot c_{B_n\times B_n}) \le \tilde{\eta}(\rho).$$

Thus we have $\rho_3 \le \rho_1$.

Remark. The main result (and its proof) remains valid in case that ρ satisfies only the following conditions

$$\rho(x,x)=0,\quad 0\le\rho(x,y)<\infty,\quad \rho(x,y)\le\rho(x,z)+\rho(z,y)$$

for all $x,y,z \in X$ if we replace $\mathrm{Lip}(f)\le a$ by $f(x)-f(y)\le a\rho(x,y)$ (X is supposed to be separable in the topology defined by the basis $\{y\in X \mid \rho(x,y)<\varepsilon,\ \rho(y,x)<\varepsilon\}$, $x\in X$, $\varepsilon>0$.

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