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8TH WINTER SCHOOL ON ABSTRACT ANALYSIS

EXTREME EXTENSIONS OF POSITIVE OPERATORS. II

BY

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The results we present here are taken from the author's papers [6] and [7], the second being a joint work with W. Thomsen (Münster). They extend and complement some results of [5] and [6] (see also [3]).

Throughout X stands for an ordered real vector space, M for its vector subspace, and Y for an order complete real vector lattice. Let $T \in L_+(M, Y)$ and put

$$E(T) = \{S \in L_+(X, Y) : \ S|M = T\}.$$

Clearly, E(T) is a convex (possibly empty) set.

We continue the discussion of the following three problems:

(A) Under what conditions $E(T) \neq \emptyset$?

(B) Under what conditions extr $E(T) \neq \emptyset$?

(C) How can the elements of extr E(T) be described?

The first to deal with (A) was Kantorovič (1937) who proved that the answer is positive provided M majorizes X. The first to deal (essentially) with (B) was Bonsall. He proved that if X has an order unit u, $M = \text{lin}\,\{u\}$ and $Y = R$, then extr $E(T) \neq \emptyset$ ([1], Theorem 3). The following more general result was proved in [5] (see also [3], Theorem 1).

THEOREM 1. extr $E(T) \neq 0$ provided M majorizes X.

Of course, the assumption that M is majorizing is not necessary. (Take $M = \{0\}$; then $S = 0$ is in extr $E(T)$.) A somewhat more complicated example shows that problems (A) and (B) are not equivalent.

EXAMPLE 1 ([7], Example 2). Let (Ω, Σ, μ) be a non-atomic probability space. Put $X = L_p(\mu)$, where $1 \leq p < \infty$, and $M = \text{lin}\{1_\Omega\}$. Define $T: M \to R$ by $T(t1_\Omega) = t$. Then $E(T)$ can be identified with the set

$$\{f \in L_q(\mu)_+ : \int_\Omega f \, d\mu = 1\},$$

where q is the exponent conjugate to p. Hence, as easily seen, extr $E(T) = \emptyset$.

Not much more seems to be known about problems (A) and (B) in the general setting we are concerned with. For a positive answer to (A) the following condition must hold: $T_e > -\infty$, where $T_e(x) \doteq \inf\{T(z): x \leq z \in M\}$. This condition is sufficient in spaces with order unit ([4], (ii)) and so in finite-dimensional spaces. Unfortunately, it does not suffice in general as shown by an der Heiden ([2], the Example). We shall give another example to the same effect.

EXAMPLE 2 ([7], Example 1). Let (Ω, Σ, μ), M and T be as in Example 1. We regard M as a subspace of $L_0(\mu)$, the vector lattice of real-valued μ-measurable functions on Ω. We have $T_e(x) = \text{ess sup } x$ for $x \in L_0(\mu)$, and so $T_e(x) > -\infty$. However, $E(T) = \emptyset$ since, by a well-known theorem of Nikodym, there are no non-zero (linear) functionals on $L_0(\mu)$ which are continuous with respect to the topology of measure convergence, and each positive functional on $L_0(\mu)$ would be continuous.

Next we turn to problem (C). From now on we assume that X is directed by its ordering. In particular, X can be a vector lattice. For $S \in L_+(X, Y)$ and $x \in X$ we put

$$S_m(x) = \inf \{ S(v) : \pm x \leq v \in X \}.$$

In case X is a vector lattice, $S_m(x) = S(|x|)$ for $x \in X$.

THEOREM 2 ([6], Theorem 2). Let $T \in L_+(M, Y)$ and $S \in E(T)$. Then $S \in \operatorname{extr} E(T)$ if and only if

$$\inf \{ S_m(x-z) : z \in M \} = 0 \quad \text{for each } x \in X.$$

For $Y = R$ this result is due (essentially) to Portenier ([9], Théorème 3.5). In case X is a vector lattice, it was obtained, independently of Portenier, by the author, Plachky and Thomsen ([4], Theorem 3; see also [3], Theorem 2). However, the first to deal with (C) seems to be Bonsall who proved a prototype of Theorem 2 for $M = \operatorname{lin}\{u\}$, where u is an order unit of X, and $Y = R$ ([1], Theorem 1).

Finally, let us mention some applications of Theorems 1 and 2.

THEOREM 3 ([7], Theorem 1). Let N be a vector subspace of Y and let $y \in Y$. Then $y \in B_N$, where B_N denotes the band generated by N, if and only if $\inf \{ |y-v| : v \in N \} = 0$.

Denote by $H(X, Y)$ the set of all $S \in L_+(X, Y)$ such that $|S(x)| = S_m(x)$ for each $x \in X$. In case X is a vector lattice, $S \in H(X, Y)$ if and only if S is a vector-lattice homomorphism.

Using Theorems 2, 3 and 1, one easily obtains

COROLLARY ([7], Theorem 3, and [5], Corollary 2). Let M be directed by its ordering and let $T \in H(M, Y)$. Then

(a) $\operatorname{extr} E(T) = E(T) \cap H(X, B_{T(M)})$.

(b) If M majorizes X, then $\operatorname{extr} E(T) = E(T) \cap H(X, Y)$; in particular, $E(T) \cap H(X, Y) \neq \emptyset$.

For M and X being vector lattices Corollary (b) has been obtained, independently and by different methods, by Luxemburg and Schep [8] (Theorems 3.1 and 4.1).

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