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A universal convex set in Euclidean space

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Professor C. Ryll-Nardzewski has raired the question whether there exists a compact convex set Q in $\mathbb{R}^3$ such that every compact convex set with non-empty interior in $\mathbb{R}^2$ is affine isomorphic to some intersection of Q with a plane.

In this note we present an example of a compact convex set Q in $\mathbb{R}^{n+2}$ $(n \geq 1)$ such that every closed convex subset of the unit ball B of $\mathbb{R}^n$ is an intersection of Q with some k-dimensional affine subspace of $\mathbb{R}^{n+2}$.

Let 2^B denote the space of all closed non-empty subsets of B endowed with the Hausdorff distance

$$\operatorname{dist}(A_1,A_2) = \max\left(\sup_{x \in A_1} d(x,A_2),\ \sup_{y \in A_2} d(y,A_1)\right)$$

where d stands for the Euclidean metric $d(x,y) = \|x-y\| = \sqrt{\langle x-y,x-y \rangle}$ in $\mathbb{R}^n$. It is well known that 2^B is compact. It is also easy to see that if $\operatorname{dist}(A_n,A_o) \to 0$ and $d(x_n,x_o) \to 0$ as $n \to \infty$ with $x_n \in A_n \in 2^B$, then $x_o \in A_o$.

Lemma. The set $\mathcal{C}$ of all convex sets in 2^B is a locally arcwise connected metric continuum.

Proof. Let a sequence A_n of elements in $\mathcal{C}$ converge to $A_o \in 2^B$ and suppose $x \in A_o$. Then clearly there exists a sequence (x_k) with $x_k \in A_k$ converging to x. This implies that if $x,y \in A_o$ then $\lambda x + (1-\lambda)y \in A_o$ for every $0 \leq \lambda \leq 1$, so A_o is convex. Thus $\mathcal{C}$ is a closed subset of 2^B, so compact.

Now we prove that $\mathcal{C}$ is locally arcwise connected. It is sufficient to show that for every different A_o, $A_1 \in \mathcal{C}$ there exists an arc A_oA_1 with diameter $\leq$ dist(A_o,A_1) (see [1], p. 242). We denote $A_t = tA_1 + (1-t)A_o = \{ty + (1-t)x : x\in A_o , y\in A_1\} \in \mathcal{T}$.

Let $x\in A_o$, $y\in A_1$ and let $x_o\in A_o$, $y_o\in A_1$ be such that $d(x,y_o)\leq$ dist (A_o,A_1) and $d(y,x_o)\leq$ dist (A_o,A_1) . For $0\leq t< <s\leq 1$ we have

$d(sy + (1-s)x,A_t)\leq d(sy + (1-s)x,\ ty + (1-t)\left[\frac{1-s}{1-t}\, x + \frac{s-t}{1-t}\, x_o\right]) =$
$= \|(s-t)(y-x_o)\| \leq |s-t|$ dist (A_o,A_1) and, analogously,
$d(ty + (1-t)x,A_s) \leq |s-t|$ dist (A_o,A_1) .

Thus for $t,s\in[0,1]$ we have

$$\text{dist } (A_t,A_s) \leq |s-t| \text{ dist } (A_o,A_1) .$$

Let $x_1,x_2\in A_o$ and $y_1,y_2\in A_1$ be such that

$$\sup_{y\in A_1} d(y,A_o) = d(y_1,A_o) = d(y_1,x_1) \quad\text{and}\quad \sup_{x\in A_o} d(x,A_1) = d(x_2,A_1) = d(x_2,y_2) .$$

Then $\sup_{x\in A_o,y\in A_1} d(ty + (1-t)x,A_o)\geq d(ty_1 + (1-t)x_1,A_o) = td(x_1,y_1)$.

For any $y\in A_1$ we have $\|(\lambda y + (1-\lambda)y_2) - x_2\|\geq\|y_2-x_2\|$ for every $\lambda\in[0,1]$, so $\langle y-y_2,\ y_2-x_2\rangle \geq 0$. For any $x\in A_o$ there exists $y_3\in A_1$ such that $d(x,y_3)\leq d(x_2,y_2)$, then $\|y_3-y_2+y_2-x_2+x_2-x\|^2 = \|y_3-x\|^2\leq \|y_2-x_2\|^2$, so $\|y_3-y_2+x_2-x\|^2 + 2\langle y_3-y_2,\ y_2-x_2\rangle \leq 2\langle y_2-x_2,\ x-x_2\rangle$. Because $\langle y_3-y_2,\ y_2-x_2\rangle \geq 0$ we have $\langle y_2-x_2,\ x-x_2\rangle \geq 0$. This implies that $\sup_{z\in A_o} d(z,A_t)\geq d(x_2,A_t) = \inf_{x\in A_o,y\in A_1} \|ty + (1-t)x - x_2\| =$
$= \inf_{x\in A_o,y\in A_1} \|t(y-y_2) + (1-t)(x-x_2) + t(y_2-x_2)\| \geq t\|y_2-x_2\|$ and
dist $(A_o,A_t)\geq \max(td(x_1,y_1),\ td(x_2,y_2)) = t$ dist (A_o,A_1) , so dist $(A_o,A_t) = t$ dist (A_o,A_1) and, analogously, dist $(A_t,A_1) = (1-t)$ dist (A_o,A_1) . Therefore for any $s,t\in[0,1]$ we obtain

$$\text{dist } (A_s,A_t) = |s-t| \text{ dist } (A_o,A_1) ,$$

so the arc $A_oA_1 = \{A_t : 0 \le t \le 1\}$ has diameter $\le$ dist (A_o, A_1).

<u>Theorem.</u> For every $n \ge 1$ there exists a compact convex set Q in $\mathbb{R}^{n+2}$ such that every closed subset of B_n can be obtained as an intersection of Q with some k-dimensional affine subspace of $\mathbb{R}^{n+2}$.

<u>Proof.</u> By the Lemma and the Peano Theorem ([1], p. 246) it follows that there exists a continuous function ψ from the interval $[0,1]$ onto $\mathcal{C}$. For $t \in [0,1]$ we define

$$C_t = \psi(t) \times \{(\cos t, \sin t)\} \subset \mathbb{R}^{n+2}$$

and put

$$Q = \operatorname{conv} \bigcup_{t \in [0,1]} C_t .$$

The set Q is compact. Indeed, let $x_k = (x_k^1, \ldots, x_k^n, \cos t_k, \sin t_k) \in Q$. Because of $\| x_k \| \le \sqrt{2}$, there exists a subsequence $x_{k'}$ of x_k converging to some $x_o = (x_o^1, \ldots, x_o^n, \cos t_o, \sin t_o) \in \in \mathbb{R}^{n+2}$. Obviously $t_{k'} \to t_o$ and $y_{k'} = (x_{k'}^1, \ldots, x_{k'}^n) \in \mathbb{R}^n$ converges to $y_o = (x_o^1, \ldots, x_o^n) \in \mathbb{R}^n$. We have $y_{k'} \in \psi(t_{k'})$ and dist $(\psi(t_k), \psi(t_o)) \to 0$. By the remark preceding Lemma this implies that $y_o \in \psi(t_o)$, so $x_o \in Q$.

Since ψ is an onto mapping, for every convex subset D of B_n there exists $t \in [0,1]$ such that $\psi(t) = D$ and for the k-dimensional affine subspace H_t of $\mathbb{R}^{n+2}$ defined as $H_t = \mathbb{R}^n \times \{(\cos t, \sin t)\}$, we have

$$Q \cap H_t = D \times \{(\cos t, \sin t)\} .$$

Indeed, let $x \in Q \cap H_t$, then there exist elements $x_i \in C_{t_i}$ and real numbers α_i, $i=1,\ldots,m$ such that $\sum \alpha_i = 1$ and $x = \sum \alpha_i x_i$. In particular $\sum \alpha_i (\cos t_i, \sin t_i) = (\cos t, \sin t)$. By the strict convexity of the unit disc in $\mathbb{R}^2$ this implies $(\cos t_i, \sin t_i) = (\cos t, \sin t)$, i.e.

$t_i = t$ for $i=1,\dots,m$. Thus $x_i \in C_t$, so $x \in D \times (\cos t, \sin t)$. Since the reverse inclusion is obvious, the proof is complete.

Let us observe that by an easy application of the Peano theorem together with some of the above arguments (for $n = 2$) the set

$$P = \bigcup_{t \in [0,1]} \{(x_1,x_2,t) : (x_1,x_2) \in \psi(t)\} \subset \mathbb{R}^3$$

satisfies the condition: Every closed convex set in $\mathbb{R}^2$ with diameter ≤ 1 can be obtained as the intersection of P with some plane (note that P is not convex).

We still do not know whether there exists a compact convex set in $\mathbb{R}^3$ with the above property.

References

[1] R. Engelking: Outline of General Topology. PWN, Warszawa 1968