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INTEGRAL REPRESENTATIONS IN CONVEX CONES

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Summary: We prove the following theorem:

Theorem 1. For every completely regular topological space E the cone $M_b^+(E)$ of bounded positive Radon measures is well capped.

This is applied to: 1) A converse theorem on integral representations.
2) A theorem on the decomposition of invariant measures into ergodic components.

Recall that a cap of a convex cone Γ is a convex compact set $K \subset \Gamma$ such that the origin belongs to K, and such that $\Gamma \setminus K$ is convex. A cone is well capped if it is the union of its caps.

The following two properties which explain the importance of well capped cones are well known:

1. Every well capped cone is the closed convex hull of its extreme rays.
2. Every closed convex subcone of a well capped cone is well capped. (cf. [1]).

Proof of theorem 1: Let $f:E \to (0, +\infty]$ be a positive function such that for each $\alpha > 0$ the set $\{x \in E: f(x) \leq \alpha\}$ is compact. Let $C_f = \{m \in M_b^+(E) : \int f dm \leq 1\}$. Then C_f is a cap in $M_b^+(E)$; this easily follows from Prohorov's theorem. We now show that every $m \in M_b^+(E)$ belongs to such a cap. There is a partition $E = N + \sum_{n \geq 1} K_n$ of the space, where the K_n are disjoint compact sets and $m(N) = 0$. Then, since $\Sigma m(K_n) < +\infty$, there exists a sequence $(\alpha_n)_{n \geq 1}$ of positive numbers with $\lim \alpha_n = +\infty$ and $\Sigma \alpha_n\, m(K_n) \leq 1$. Let $f(x) = \alpha_n$ on K_n, $f(x) = +\infty$ on N. Then $\{x : f(x) \leq \alpha\} = \bigcup_{\alpha_n \leq \alpha} K_n$ is compact and $\int f dm \leq 1$, i.e. $m \in C_f$.

Theorem 2. Let Γ be a closed convex cone in a quasi-complete locally convex hausdorff space. Assume Γ has a bounded base B and assume every point of B is the resultant of a unique Radon probability measure on the extreme points of B. Then Γ is well capped.

Proof. Let E be the set of extreme points of B. The space being quasi-complete it can be shown that the map $r : m \to \int x\, dm(x)$ from $M_b^+(E)$ to Γ is well defined. It is continuous in the weak topology and by hypothesis bijective. Moreover, it can be shown that the restriction of r to a cap C_f (notation of proof of theorem 1) is continuous. Thus $r(C_f)$ is a cap in Γ and Γ is the union of such caps.

Theorem 3. Let E be a completely regular Souslin space. Let A be a closed convex subset of $M_+^1(E)$.

Then 1) Every point $a \in A$ is the resultant of a Radon probability on the set $\mathcal{E}(A)$ of extreme points of A.

2) This measure is uniquely determined for each $a \in A$ if and only if A is a simplex (i.e. the cone $\Gamma = \bigcup_{\lambda \geq 0} \lambda A$ is a lattice).

Proof. This will follow from a general theorem on integral representations ([2] Corollaire 4) is we prove that Γ has the following two properties:

a) Γ is the union of metrizable caps.

b) The closed convex hull of each compact subset of Γ is compact.

It suffices to prove these properties for the cone $M_b^+(E)$ instead of Γ. Now a) follows from theorem 1 and from the fact that $M_b^+(E)$ is a Souslin space (in the topology $\sigma(M_b^+, C_b)$; cf. [3]), which implies that every compact subset of $M_b^+(E)$ is metrizable.

In order to prove b) it is sufficient to prove that for every compact space K, every continuous map $t \to \mu_t$ from K to $M_b^+(E)$ and every Radon measure m on K, there exists $\mu \in M_b^+(E)$ such that

$$(1) \qquad \mu(\varphi) = \int \mu_t(\varphi)\, dm(t) \qquad \forall \varphi \in C_b(E).$$

In order to prove this we define a linear form μ on $C_b(E)$ by the formula (1). Then μ is clearly a Daniell integral on $C_b(E)$, and so, by Daniell's theorem there exists a bounded measure P on the smallest -al bra, rendering tl e functions in $C_b(E)$ measurable, such that

$\mu(\varphi) = \int \varphi dP$. Now E being a Souslin space this σ-algebra coincides with the Borel σ-algebra of E, and P is a Radon measure. Thus we may identify P and μ and we are done.

Application to invariant measures: - Let E be a completely regular Souslin space and let G be a group of homeomorphisms of E. Then every G-invariant probability measure $\underline{\mu}$ on E has a unique decomposition

$$(2) \qquad \underline{\mu} = \int \mu \, dm(\mu)$$

in ergodic components.

Proof. It suffices to apply the previous theorem to the set A of G-invariant probability measures. Then $\Gamma = \bigcup_{\lambda > 0} \lambda A$ is the set of all G-invariant bounded measures. Since the supremum in $M_b^+(E)$ of two elements of Γ again belongs to Γ it follows that Γ is a lattice, and theorem 3 may be applied.

Remark (2) is equivalent to

$$\underline{\mu}(B) = \int \mu(B) dm(\mu)$$

for all Borel sets B.

In this form the result could possible be extended, with the help of the methods of F. Topsoe, to the case where E is a, not necessarily completely regular, Souslin space.

Example (cf. K. Gawędzki): - $E = S'(\mathbb{R}^d)$ G the Euclidean motion group.

[1] G. Choquet, Lectures on Analysis (Benjamin).

[2] E.G.F. Thomas, Représentations intégrales dans les cônes convexes conucléaires et applications. Seminaire Choquet. (Initiation à l'analyse) 17e annee, 1977/78, No 9.

[3] N. Bourbaki, Integration, chapitre IX.

[4] F. Topsoe, Topology and measure, Springer lecture notes in mathematics 133 (1970).

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