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THE INNER STRUCTURE OF REAL EXTENSORS IN UNIFORM SPACES

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We refer to [I] for basic concepts pertaining to uniform spaces. Under a space we always mean a Hausdorff uniform space. Recall that a uniform space X is called a real extensor, (RE), if every uniformly continuous real-valued function on a subspace of X extends to a uniformly continuous function on X. The aim of this note is to give the full description of this important property by means of uniform covers only.

Several sufficient conditions have been known for a space to be RE. J.Isbell [I] proved that locally fine spaces have this property, this result was generalized by Z.Frolík [F] for sub-inversion-closed spaces, and finally Z.Frolík, J.Pelant and the author [FPV] proved that even each so called hedgehog-fine space enjoys the property RE. Recall that the space is called hedgehog-fine, if countable uniformly discrete unions of finite uniformly discrete families are uniformly discrete.

All the three preceding sufficient conditions are coreflective in uniform spaces (closed under the formation of uniform sums and quotients) and it can be proved (see [FPV]) that the class of all hedgehog-fine spaces is the largest coreflective class (in uniform spaces) contained in the class RE.

However this sufficient condition for a space to be RE is still far from being necessary. For instance one can easily see that every zerodimensional uniform space has the property RE, but, of course, need not be hedgehog-fine.

If we want to characterize the uniform covers of real extensors, the following "unstable" property comes in in a rather essential way:

Definition: Let X be a uniform space, Y its subspace. A uniform cover $\mathcal{U}$ of Y is said to be perfectly refinable in X, if for each finite pseudometric ρ on $\mathcal{U}$ there is a uniform cover $\mathcal{V}$ of X such that $St^k\mathcal{V}$ restricted to Y refines the cover

$$\left\{ \bigcup_{\rho(U,V)\leq k} V \; ; U \in \mathcal{U} \right\}$$

for all $k \in \omega$.

If X = Y, we simply call $\mathcal{U}$ perfectly refinable.

Here $St^0\mathcal{V} = \mathcal{V}$ and for $k \geq 1$ $St^k\mathcal{V}$ means the cover $\{St^k(x,\mathcal{V}); x \in X\}$, where $St^k(x,\mathcal{V}) = \{y \in X$; there exist $V_1,\dots,V_k \in \mathcal{V}$ with $x \in V_1$, $y \in V_k$ and $V_{i-1} \cap V_i \neq \emptyset$ for all i=2,...

Simple examples show that this property is really very "unstable". The meet of two perfectly refinable covers need not be perfectly refinable, also a perfectly refinable uniform cover of a space need not posess a perfectly refinable uniform star-refinement.

The following nontrivial theorem is a recent result of D.Preiss and the author. Recall that a uniform cover $\mathcal{U}$ is called finite-dimensional, if there is a natural number n such that each point is contained in at most n members of $\mathcal{U}$.

Theorem: Let X be a uniform space, Y its subspace. The following conditions are equivalent:

(1) Every uniformly continuous real-valued function defined on any subspace of Y has a uniformly continuous extension over X.

(2) Every countable finite-dimensional uniform cover of Y is perfectly refinable in X.

The proof can be found in [PV]. The necessity of condition (2) is proved by careful handling with uniform covers of Euclidian spaces, the proof of its sufficiency uses rather deep theorems on in-between type studied systematically in [PV].

If we put [illegible] in the Theorem, we obtain the promised characterization of [illegible] - spaces:

Corollary: A uniform space X is a real extensor if and only if every countable finite-dimensional uniform cover of X is perfectly refinable.

At the end we show how to apply this result to the construction of some RE-uniformities. It is proved in [FPV] that if R' is an RE-uniformity on the real line finer than the usual metrizable uniformity R such that R' is a value of R under some coreflector in the category of uniform spaces, then R' is finer than the topologically fine uniformity $t_f R$ (having for basis all open covers of R). However it was a problem, whether (omitting the last assumption) one can find some RE-uniformity on the real line finer than R and strictly coarser than $t_f R$. The following example gives even infinitely many such uniformities.

Example: Let D be an infinite uniformly discrete subset of R. Let ν be the family of all open covers $\mathcal{U}$ of R for which there is $\varepsilon > 0$ such that for each $d \in D$ one can find $U \in \mathcal{U}$ with $(d-\varepsilon, d+\varepsilon) \subset U$. Then ν is a basis of an RE-uniformity which is finer than R and strictly coarser than $t_f R$.

Sketch of the proof: The family ν is obviously closed under meets. Moreover each cover from ν can be refined by a cover $\mathcal{U} = \{(a_n, b_n) ; n \in Z\}$ such that $a_n < b_{n-1} < a_{n+1}$ for all

$n \in Z$ and, for some $\delta > 0$ and each $d \in D$ there is a unique integer n with $b_{n-1} \leqslant d-\delta < d+\delta \leqslant a_{n+1}$. It is not much difficult to show that $\mathcal{U}$ is perfectly refinable in $\mathcal{V}$. (In particular $\mathcal{U}$ has a star-refinement in $\mathcal{V}$.) Therefore using the Corollary $\mathcal{V}$ is a basis of an RE-uniformity. The remaining assertions are obvious.

Observe that for each open cover $\mathcal{V}$ of R, which is not (metrically) uniform, we can find (by induction) an infinite uniformly discrete subset D of R with the property that for any $\varepsilon > 0$ there is $d \in D$ such that no $V \in \mathcal{V}$ contains the interval $(d-\varepsilon, d+\varepsilon)$. The preceding Example constructs an RE-uniformity finer than R, coarser than $t_f R$ and such that $\mathcal{V}$ is not uniform.

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