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We consider the lattice of uniformities on a countable set, say on ω (w.r.t. the order "$\mathcal{a} < \mathcal{b}$ iff $\mathcal{a}$ is finer than $\mathcal{b}$"). A uniformity $\mathcal{a}$ is an atom in this lattice iff the only uniformity strictly finer than $\mathcal{a}$ is uniformly discrete one. The investigation of non-0-dimensional atoms was initiated by the fact that first constructions [PR], [S] led to 0-dimensional atoms (a uniformity is 0-dimensional if it admits a base consisting of partitions). E.g. in [PR] a complete description of atoms inducing non-discrete proximities is given; all these atoms are 0-dimensional. The existence of non-zero dimensional atoms was established in [RR] under the CH.

If an atom $\mathcal{a}$ induces the discrete proximity then there exists an ultrafilter $\mathcal{F}$ with $\mathcal{a} < \mathcal{U}_{\mathcal{F}}$ where $\mathcal{U}_{\mathcal{F}}$ is the uniformity consisting of all covers $\mathcal{C}$ with $\mathcal{C} \cap \mathcal{F} \neq \emptyset$. [PR]. The problem to describe all non-zero dimensional atoms or at least to characterize those ultrafilters $\mathcal{F}$ with $\mathcal{a} < \mathcal{U}_{\mathcal{F}}$ for some non-zero dimensional atom $\mathcal{a}$ seems to be difficult. By [S], each of these $\mathcal{F}$'s is non-rare, that is, it admits a finite-to-one map $q : \omega \to \omega$ such that $\mathcal{G} = q\mathcal{F}$ is non-equivalent to $\mathcal{F}$. So we considered an easier problem: characterize those $\mathcal{G}$ for which there exists $q, \mathcal{F}, \mathcal{a}$ as above. The solution is given by the theorem below which solves also the problem of the existence of two uniformities $\mathcal{U}_1, \mathcal{U}_2$ with dist $\mathcal{U}_1 \wedge$ dist $\mathcal{U}_2 \neq$ dist $(\mathcal{U}_1 \wedge \mathcal{U}_2)$; here dist $\mathcal{U}$, the distality of $\mathcal{U}$, is the uniformity generated by $\mathcal{U}$-covers of finite order.

Theorem : For every ultrafilter $\mathcal{G}$ on ω there exist a finite-to-one map $q: \omega \to \omega$ an ultrafilter $\mathcal{F}$ on ω with $q\mathcal{F} = \mathcal{G}$ and an uncountable family $\{a_i\}_i$ of non-zero dimensional atoms such that $a_i < u_{\mathcal{F}}$ for every i. The atoms a_i have the same distality. Thus, $\operatorname{dist} a_{i_1} \wedge \operatorname{dist} a_{i_2} \neq$ $\neq \operatorname{dist}(a_{i_1} \wedge a_{i_2})$ for any two of them.

References :

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[S] P. Simon: Uniform Atoms on ω, Seminar Uniform Spaces 1975 - 1976, MÚ ČSV Prague 1976, p. 7 - 35