David Preiss

Invalid Vitali theorems

In: Zdeněk Frolík (ed.): Abstracta. 7th Winter School on Abstract Analysis. Czechoslovak Academy of Sciences, Praha, 1979. pp. 58--60.

Persistent URL: http://dml.cz/dmlcz/701149

SEVENTH WINTER SCHOOL OF ABSTRACT ANALYSIS (1979)

INVALID VITALI THEOREMS

D. Preiss

Vitali type covering theorems in finite dimensional Banach spaces hold (under some regularity assumptions on the considered covers) for arbitrary measures (see [M]). If we drop the assumption of finite dimensionality the situation becomes different. By a result of Davies [D] there exist distinct probability measures on a metric space which agree on all balls. Although this particular behaviour is not possible in the case of Hilbert spaces, it was shown in [P] that Vitali Theorem does not hold for centered balls and Gaussian measures. The following result shows that even the Density Theorem does not hold in infinitely dimensional Hilbert spaces.

Theorem. Let H be a separable infinitely dimensional real Hilbert space. Then there is a finite measure u on the Borel σ-algebra of H and a compact set $C \subset H$ such that $u(C) > 0$ and

$$\lim_{r \to 0_+} \frac{u(C \cap B(x,r))}{u(B(x,r))} = 0 \quad \text{for each } x \in C.$$

Proof. By induction one easily defines a sequence $\{a_k\}$ of positive numbers and a sequence $\{N_k\}$ of natural numbers such that

$$\sum_{k=1}^{\infty} a_k N_1 \ldots N_k < \infty \quad \text{and} \quad \lim_{k \to \infty} a_k N_1 \ldots N_{k+1} = \infty .$$

Let S be the set of all finite sequences $(z_1, \ldots, z_k)$ of natural numbers such that $z_i \le N_i$ and let Z be the set of all infinite sequences $(z_1, \ldots)$ of natural numbers such that $z_i \le N_i$.

For each $z = (z_1, \ldots, z_k) \in S$ choose $h(z) \in H$ such that

$\|h(z)\|^2 = 2^{-k}$ and $h(y), h(z)$ are orthogonal whenever $y,z \in S$, $y \neq z$.

Put

$$g(z) = \sum_{j=1}^{k} h(z_1,\dots,z_j) \qquad \text{for } z=(z_1,\dots,z_k)\in S\,,$$

$$f(z) = \sum_{j=1}^{\infty} h(z_1,\dots,z_j) \qquad \text{for } z=(z_1,\dots)\in Z\,.$$

Note that $\|f(y) - f(z)\|^2 = 2^{-k+2}$ if $y,z\in Z$, $y \neq z$ and k is the least natural number such that $z_k \neq y_k$ and $\|f(z) - g(z_1,\dots,z_k)\|^2 = 2^{-k}$ for each $z\in Z$ and natural k.

The set Z considered as a product of finite topological spaces is a compact metrizable space. Let v be the product of measures v_j on the sets $\{1,\dots,N_j\}$, where $v_j(n) = (N_j)^{-1}$.

Put $u = f(v) + \sum_{(z_1,\dots,z_k)\in S} a_k\, \varepsilon_{g(z_1,\dots,z_k)}$,where $f(v)$ is the image measure and ε_x is the Dirac measure at x.

If $C = f(Z)$, $z\in Z$, $x = f(z)$ and $2^{-k}\leq r^2 < 2^{-k+1}$ then $u(B(x,r)\cap C) = v\{y\in Z;\ y_i = z_i \text{ for } i=1,\dots,k+1\} = (N_1\dots N_{k+1})^{-1}$ and $u(B(x,r)) \geq a_k$, since $g(z_1,\dots,z_k)\in B(x,r)$. Thus

$$\frac{u(B(x,r)\cap C)}{u(B(x,r))} \leq (a_k N_1\dots N_{k+1})^{-1}\,.$$

Remark. If we construct the sequences $\{a_k\}, \{N_k\}$ so that $\sum_{k=1}^{\infty} a_k N_1\dots N_k < 1$, then the measure $w = u - 2f(v)$ has the following properties

(i) $w(H) < 0$

(ii) for each $x\in H$ there is $r(x)>0$ such that $w(B(x,r)) \geq 0$ for each positive $r<r(x)$.

This example should be compared with a recent result of Christensen [C]: If u is a measure on H such that for each $x\in H$

there exists $r(x) > 0$ such that u vanishes on all balls contained in the ball with center x and radius r(x), then u vanishes identically.

References.

[C] Christensen, J.P.R.: The Small Ball Theorem for Hilbert Spaces. Math.Ann. 237,27[illegible]-276 (1978)

[D] Davies,R.O.: Measures not approximable or not specifiable by means of balls. Mathematika 18,157-160 (1971)

[M] Morse, A.P.: Perfect blankets. Trans.Amer.Math.Soc. 61,418-442 (1947)

[P] Preiss,D.: Gaussian measures and covering theorems, Comment. Math.Univ.Carolin. 20,95-99 (1979)