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On the existence of weak P*-points in compact* F*-spaces*

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All spaces are completely regular and X* denotes βX-X.

The point $x \in X$ is called a P-point whenever $x \notin \overline{F}$ for each F_σ F of X which does not contain x. It is known that ω^* contains P-points under CH (cf. RUDIN [ ]); however, SHELAH (see [M] or [W]) showed that it is consistent with the usual axioms of set theory that there are no P-points in ω^*. The point $x \in X$ is called a weak P-point whenever $x \notin \overline{F}$ for each countable $F \subset X-\{x\}$. Clearly each P-point is a weak P-point. Recently KUNEN [K] showed that there are 2^{2^ω} points in ω^* which are weak P-points. We have a generalization of this result.

0. THEOREM: *Let* X *be a compact infinite* F*-space without isolated points of weight* 2^ω *in which each nonempty* G_δ *has nonempty interior. Then there are* 2^{2^ω} *points in* X *which are weak* P*-points but not* P*-points.*

The condition that each nonempty G_δ has nonempty interior is essential of course, since no separable space without isolated points can have weak P-points (we don't know whether the theorem is true for compact nowhere separable F-spaces). There are two ways of trying to generalize this result:

question (1): can we delete "of weight 2^{ω}" from the hypotheses?

question (2): can we delete "F-space" from the hypotheses?

In our opinion, question (1) is easier than question (2). Under CH both questions have a positive answer. For question (2) this is trivial (let us notice that question (2) and the observation that CH implies a positive answer are due to KUNEN). The proof that CH implies a positive answer to question (1) is not trivial. In fact it is quite surprising that CH is used to find special points in spaces of arbitrarily large weight. Usually CH is used to build by induction an ultrafilter in ω_1 steps, killing at each stage of the induction an undesired set. We use CH in a completely different form. Let (*) denote the innocent statement that there is a compactification $\gamma\omega$ of ω such that $\gamma\omega-\omega$ is ccc but not separable. We show that CH implies (*) and that (*) gives a positive answer to question (1):

1. THEOREM: *Assume (*) and let* X *be a compact infinite* F*-space in which each nonempty* G_δ *without isolated points has nonempty interior. Then there are* $2^{2^{\omega}}$ *weak* P*-points in* X *which are not* P*-points.*

The proof that CH implies (*) is easy. By TALL [T, Ex.7.5] the Stone space of the boolean algebra of Lebesgue measurable sets of [0,1] modulo the nullsets is a compact extremally disconnected ccc nonseparable space of weight 2^{ω}. Under CH, each compact space of weight at most 2^{ω} is a continuous image of ω^*, or, equivalently, is the remainder of some compactification of ω (cf. PAROVIČENKO [P]). Hence CH implies (*).

I conjecture that (*) is true in ZFC, but would not be too much surprised if (*) turned out to be consistently false. If $\neg$(*) is consistent it would be an interesting new axiom, since it gives tremendous control on the subalgebras of $P(\omega)$/finite. The easiest way of answering the question whether (*) is true would be to construct a compact ccc nonseparable space of weight ω_1, since each compact space of weight ω_1 is the remainder of some compactification of ω([P]). However, under MA+$\neg$CH, each compact ccc space of weight less than 2^{ω} is separable ([T, Theorem 1.4(a)]), which blocks this attempt (this was brought to my attention by Eric van Douwen).

Let us finally notice that I have also shown that Theorem 1 is true if $2^{\omega} = 2^{\omega_1}$. Consequently, Theorem 1 is true in ZFC if $\neg$(*) implies $2^{\omega} = 2^{\omega_1}$.

The proofs of our claims will appear in [vM].

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