Zbigniew Lipecki
Extreme extensions of positive operators

In: Zdeněk Frolík (ed.): Abstracta. 6th Winter School on Abstract Analysis. Czechoslovak Academy of Sciences, Praha, 1978. pp. 63--66.

Persistent URL: http://dml.cz/dmlcz/701123

6TH WINTER SCHOOL

EXTREME EXTENSIONS OF POSITIVE OPERATORS

BY

Z. LIPECKI

The results we present here are taken from the author's papers [2] and [3], the first one being a joint work with D. Plachky and W. Thomsen (Münster).

Throughout we adhere to the terminology of Schaefer's monograph [6]. We use the following notation. X stands for an ordered real vector space, M for its vector subspace and Y for an order complete real vector lattice. Given $T \in L_+(M, Y)$ (i.e. a positive linear operator from M into Y), we put $E(T) = \{S \in L_+(X, Y): S|M = T\}$. We shall be concerned with the extreme points of E(T).

THEOREM 1 ([3], Theorem 1). If M is a majorizing (i.e. cofinal) subspace of X, then extr $E(T) \neq \emptyset$.

This is an improvement of a classical result of L. V. Kantorovič ([7], Theorem X.3.1, or [2], Theorem 1) who proved that $E(T) \neq \emptyset$.

THEOREM 2 ([2], Theorem 3). Suppose X is a vector lattice and $S \in E(T)$. Then $S \in$ extr E(T) if and only if $\inf\{S(|x - z|): z \in M\} = 0$ for each $x \in X$.

We shall give a number of applications of Theorems 1 and 2.

In the first two corollaries X is assumed to be a vector lattice and M its vector sublattice. We denote by $H(M, Y)$ the set of all lattice homomorphisms of M into Y.

COROLLARY 1 ([3], Theorem 2). Suppose $T \in H(M, Y)$.

(a) $\operatorname{extr} E(T) \subset H(X, Y)$.

(b) If $\inf \{|y - T(z)| : z \in M\} = 0$ for each $y \in Y$, then $E(T) \cap H(X, Y) \subset \operatorname{extr} E(T)$.

COROLLARY 2 ([3], Corollary 2). If M is majorizing, then any lattice homomorphism $T: M \to Y$ extends to a lattice homomorphism $S: X \to Y$.

As another application we shall give a characterization of the extreme points of certain sets of operators between vector lattices of measurable functions. Let $(\Omega_i, \Sigma_i, \mu_i)$, where $i = 1, 2$, be positive finite measure spaces. Denote by $L_0(\mu_i)$ the (order complete) vector lattice of (μ_i-equivalence classes of) real-valued measurable functions on Ω_i and by $s(\mu_i)$ its vector sublattice consisting of all simple functions. The following corollary generalizes Propositions I.4.3 and 4 in [6] on stochastic matrices. It is also akin to some results of Phelps ([4], Theorem 2.2) and Iwanik ([1], Lemma 2 and Proposition 2).

COROLLARY 3 ([3], Theorem 3). Let X be a vector sublattice of $L_0(\mu_1)$ containing $s(\mu_1)$ and let Y be an order complete vector sublattice of $L_0(\mu_2)$. Suppose that given $x \in X$, there exist $x_n \in s(\mu_1)$, $v \in X_+$ and $\varepsilon_n \in R_+$ with $|x - x_n| \leq \varepsilon_n v$ and $\varepsilon_n \downarrow 0$. Then for each $S \in L_+(X, Y)$ with $S 1_{\Omega_1} = 1_{\Omega_2}$ the following three conditions are equivalent:

(i) $S \in \text{extr}\,\{T \in L_+(X, Y):\ T1_{\Omega_1} = 1_{\Omega_2}\}$.

(ii) S takes characteristic functions into characteristic functions.

(iii) $S \in H(X, Y)$.

It can be proved that the assumptions of Corollary 3 are satisfied for $X = L_{p_1}(\mu_1)$, $Y = L_{p_2}(\mu_2)$, where $0 \leqslant p_1, p_2 \leqslant \infty$.

Finally, we shall apply Theorem 2 to additive set functions. Let $\mathcal{R}$ and $\mathcal{S}$ be rings of sets with $\mathcal{R} \subset \mathcal{S}$. We say that $\mu : \mathcal{R} \to Y$ is a content provided it is additive and $\mu(C) \geqslant 0$ for all $C \in \mathcal{R}$. Given a content $\mu : \mathcal{R} \to Y$, we denote by $E(\mu)$ the set of all contents on $\mathcal{S}$ extending μ. The following is a generalization of a theorem due to Plachky ([5], Theorem 1).

COROLLARY 4 ([2], Theorem 4). Suppose $\nu \in E(\mu)$. Then $\nu \in \text{extr}\, E(\mu)$ if and only if $\inf\{\nu(A \triangle C):\ C \in \mathcal{R}\} = 0$ for each $A \in \mathcal{S}$.

REFERENCES

[1] A. Iwanik, Pointwise induced operators on L_p-spaces, Proc. Amer. Math. Soc. 58 (1976), p. 173-178.

[2] Z. Lipecki, D. Plachky and W. Thomsen, Extensions of positive operators and extreme points. I (submitted).

[3] Z. Lipecki, Extensions of positive operators and extreme points. II, Coll. Math. (to appear).

[4] R. R. Phelps, Extreme positive operators and homomorphisms, Trans. Amer. Math. Soc. 108 (1963), p. 265-274.

[5] D. Plachky, Extremal and monogenic additive set functions, Proc. Amer. Math. Soc. 54 (1976), p. 193-196.

[6] H. H. Schaefer, Banach lattices and positive operators, Berlin-Heidelberg-New York 1974.

[7] B. Z. Vulikh, Introduction to the theory of partially ordered vector spaces, Groningen 1967.