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by

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Throughout, $|S|$ denotes the cardinality of the set S, $\mathcal{P}(S)$ the power set of S, $[S]^{\nu} = \{X \subset S : |X| = \nu\}$, and $V=L$ denotes Gödel's axiom of construotability. Small greak letters denote ordinals, with $\varkappa, \mu$ always denoteing infinite cardinals and λ, ν any (finite or infinite) cardinals. The following corollary follows from our Theorem 3.

COROLLARY. Let F be a family of σ-fields of subsets of the real line S, such that $[S]^1 \subset A$ and $A \neq \mathcal{P}(S)$ for every $A \in F$. Then

a) $|F| < \omega$ implies $\bigcup F \neq \mathcal{P}(S)$;

b) If $2^{\omega} = \omega_1$ then $|F| \leqslant \omega$ implies $\bigcup F \neq \mathcal{P}(S)$;

c) If $V = L$ then $|F| \leq \omega_1$ implies $\bigcup F \neq \mathcal{P}(S)$.

The Corollary can be strenghtened even under weaker set theoretical assumption (see Theorem 3). In the case of an additional assumption that on each $A \in F$ it is possible to define a non-trivial measure (or even that A satisfies only certain chain condition), the Corollary has been known. In that case, a) is due to Ulam (see [1]), b) is a theorem of Alaoglu - Erdös (see [1] and also [4] and [3]), and c) is a theorem of Prikry (see [4], for generalizations see [3], for strenghtenings and further generalizations see [6]. In case on each $A \in F$ it is possible to define a non-trivial two-valued measure, c) is a theorem of Jensen (see [0]).

The strongest and the most general results connected with a problem of Ulam on families of measures (see problem 81 of [2] and also [8]) have been recently obtained by Taylor in [6]. The main subject of this note is a generalization of two theorems of Taylor in [6].

If $Q \subset \mathcal{P}(S)$ then we define $I(Q) = \{X \in Q : \mathcal{P}(X) \subset Q\}$. Q will be called μ-complete iff for every $X \subset Q$ such that $|X| < \mu$ we have $\bigcup X \in Q$. Remark that if Q is μ-complete then $I(Q)$ is a μ-complete ideal on S. Q will be called non-trivial iff $[S]^1 \subset Q$ and $Q \neq \mathcal{P}(S)$.

A family $F \subset \mathcal{P}(\mathcal{P}(S))$ will be called ν - saturated w.r.t. I, where I is an ideal on S such that $I \subset \bigcap\{I(A) : A \in F\}$, iff for every collection $\{X_\alpha : \alpha < \nu\} \subset \mathcal{P}(S) - \bigcup F$ there exists $\{\alpha,\beta\} \in [\nu]^2$ such that $X_\alpha \cap X_\beta \notin I$.

A family $F \subset \mathcal{P}(\mathcal{P}(S))$ will be called ν-saturated iff F is ν-saturated w.r.t. $I = \bigcap\{I(A) : A \in F\}$.

The following two definitions are central for the considerations of this note.

If $Q \subset \mathcal{P}(\mathcal{P}(\varkappa))$ then the symbol

$$\text{"}\langle \varkappa : \lambda, \mu \rangle \xrightarrow{Q} \nu\text{"}$$

denotes the following assertion.

If $F \subset Q$, $|F| \leq \lambda$ and I(A) is μ-complete for every $A \in F$ then F is not ν-saturated.

If $Q \subset \mathcal{P}(\mathcal{P}(\varkappa))$ and I is an ideal on $\varkappa$ (we do not exclude the case $I = \{\emptyset\}$) then the symbol

$$\text{"}\langle \varkappa : \lambda, \mu \rangle \xrightarrow{Q} \langle \nu, I \rangle\text{"}$$

denotes the following assertion.

If $F \subset Q$, $|F| \leq \lambda$, $I \subset \bigcap\{I(A) : A \in F\}$ and I(B) is μ-complete for every $B \in F$ then F is not ν-saturated w.r.t. I.

In case Q is a set of all non-trivial ideals on $\varkappa$ the notation $\langle \varkappa : \lambda, \mu \rangle \xrightarrow{Q} \nu$ was introduced by Taylor in [6]. If Q is a set of all non-trivial ideals on $\varkappa$ then instead of $\langle \varkappa : \lambda, \mu \rangle \xrightarrow{Q} \nu$ and $\langle \varkappa : \lambda, \mu \rangle \xrightarrow{Q} \langle \nu, I \rangle$ we will write $\langle \varkappa : \lambda, \mu \rangle \longrightarrow \nu$ and $\langle \varkappa : \lambda, \mu \rangle \longrightarrow \langle \nu, I \rangle$, respectively (i.e. we suppress the superscript Q in this case).

For a fixed cardinal $\varkappa$ we define

$R = \{A \subset \mathcal{P}(\varkappa) : A \text{ is non-trivial and } \forall (a \in A) \forall (b \in A) (a \cap b \in A \text{ and } a - b \in A)\}$

We have the following theorem.

THEOREM 1. Assume $\lambda \leq \nu \geq \omega$. Then we have

a) If I is a $(\lambda + \omega)$-complete ideal on $\varkappa$ then

$\langle \varkappa : \lambda, \mu \rangle \xrightarrow{R} \langle \nu, I \rangle$ iff $\langle \varkappa : \lambda, \mu \rangle \longrightarrow \langle \nu, I \rangle$.

a') If $\lambda \leq \mu$ then

$\langle \varkappa : \lambda, \mu \rangle \xrightarrow{R} \nu$ iff $\langle \varkappa : \lambda, \mu \rangle \longrightarrow \nu$.

From Theorem 1 we have in particular the following result:

$\langle \omega_1 : \omega_1, \omega_1 \rangle \longrightarrow \omega_1$ iff $\langle \omega_1 : \omega_1, \omega_1 \rangle \xrightarrow{R} \omega_1$.

This (and also our Theorem 3) should be compared with the comments of the authors of [2] on the problem 81 of Ulam (see also [8]).

With the help of Theorem 1 we will generalize the following results of Taylor (Theorem 2.2 and Theorem 4.4. of [6]). We fomulate them in a little more general form, which easily follows from the original one.

THEOREM 2 (TAYLOR)

a) Assume $\nu \geq \lambda^+ + \omega$, $\mu \geq \lambda^+ + \omega$, $\lambda < \varkappa$ and I is a $(\lambda^+ + \omega)$-complete ideal on $\varkappa$. Then

$$\langle \varkappa : \lambda, \mu \rangle \rightarrow \langle \nu, I \rangle \quad \text{iff} \quad \langle \varkappa : 1, \mu \rangle \rightarrow \langle \nu, I \rangle$$

b) $\langle \omega_1 : \omega_1, \omega_1 \rangle \rightarrow \omega_2$ iff $\langle \omega_1 : 1, \omega_1 \rangle \rightarrow \langle \omega_2, [\omega_1]^{<\omega_1} \rangle$

Recall that the above theorem of Taylor is a strenghtening and a generalization of results of Ulam, Alaoglu - Erdös (see [1]), Jensen (see [0]), Prikry (see [4]) and of the present author (see [3]). By Theorem 1 and Theorem 2 we have the following generalization of Theorem 2.

THEOREM 3. a) Assume I is a $(\lambda^+ + \omega)$- complete ideal on $\varkappa$ and $\nu \geq (\lambda^+ + \omega)$, $\mu \geq \lambda^+ + \omega$, $\lambda < \varkappa$. Then

$$\langle \varkappa : \lambda, \mu \rangle \xrightarrow{R} \langle \nu, I \rangle \quad \text{iff} \quad \langle \varkappa : 1, \mu \rangle \longrightarrow \langle \nu, I \rangle .$$

b) $\langle \omega_1 : \omega_1, \omega_1 \rangle \xrightarrow{R} \omega_2$ iff $\langle \omega_1 : 1, \omega_1 \rangle \rightarrow \langle \omega_2, [\omega_1]^{<\omega_1} \rangle$.

Remark that if we replace R by $R_0 \subset R$, where R_0 is a collection of families of subset of $\varkappa$, satisfying certain natural chain conditions, then Theorem 3 becomes a known result which easily follows directly from Theorem 2 (see Corollary 4.13 of [6], compare also [3] and [4]).

To see for which $\varkappa, \lambda, \mu, \nu$ Theorem 3 works, recall the following well known facts. $\langle \varkappa^+ : 1, \varkappa^+ \rangle \rightarrow \varkappa^+$ and $\langle 2^\varkappa : 1, \varkappa^+ \rangle \rightarrow \omega$ holds for every $\varkappa$ (see [7]). $\langle \varkappa : 1, \mu \rangle \rightarrow \mu$ holds for every $\varkappa$ which is less than the first weakly innaccessible cardinal and every $\mu \leq \varkappa$ (easily follows from the first previous relations). $\langle \varkappa : 1, \omega_1 \rangle \rightarrow \omega$ holds for every $\varkappa$ which is less than the first strongly innaccessible cardinal (see [7]). By results of Tarski and Solovay the relations holds if $\varkappa$ is even larger. It is also well known that the axiom of constructability (V = L) implies $\langle \omega_1 : 1, \omega_1 \rangle \rightarrow \langle \omega_2, [\omega_1]^{<\omega_1} \rangle$ (see [5]).

The elementary proof of Theorem 1 will be submitted elsewhere.

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