Robert Frič

Sequential completeness versus Čech - completeness

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SEQUENTIAL COMPLETENESS VERSUS ČECH-COMPLETENESS

by

R.Frič, Žilina

Detail information about sequentially complete spaces can be found in [2]. Concerning Čech-complete spaces we refer to [1]. In this short note we present two examples showing that these two topological properties are independent. All spaces are assumed to be completely regular.

Let X be a space and βX its Čech-Stone compactification. Recall that X is <u>sequentially complete</u> iff it is sequentially closed in βX (i.e. no sequence in X converges to a point in $\beta X-X$) and X is <u>Čech-complete</u> iff X is a G_δ-set in βX (i.e. X is an intersection of countably many open sets in βX).

Example 1. Let Q be the space of all rational numbers. Then:

(i) Q is sequentially complete.

(ii) Q fails to be Čech-complete.

Proof. (i) follows from the realcompactness of Q. The direct proof is, however, straightforward.

(ii) is well-known.

Example 2. Consider the set $X = ((\omega_0+1)\times(\omega_1+1)) - \{(\omega_0,\omega_1)\}$ equipped with the following topology: all points (ξ,η), $\xi\in\omega_0$, $\eta\in\omega_1$, are isolated; for each (ω_0,η), sets $\{(\omega_0,\eta)\}\cup\cup\{(\xi,\eta) : \xi\in\omega_0$ for all but finitely many $\xi\}$ form a local base at (ω_0,η); for each (ξ,ω_1), sets $\{(\xi,\omega_1)\}\cup\{(\xi,\eta) : \eta\in\omega_1$ for all but finitely many $\eta\}$ form a local base at (ξ,ω_1). Then:

(i) X is Čech-complete.

(ii) X fails to be sequentially complete.

Proof. (i) follows from the local compactness of X.

(ii) let $f\in C^*(X)$. Then for each $\xi\in\omega_0$ there is $\eta(\xi)$

such that for each $\eta \geq \eta(\xi)$ we have $f((\xi,\eta)) = f((\xi,\eta(\xi))) = f(\xi)$. Put $\eta(f) = \sup\{\eta(\xi) : \xi \in \omega_0\}$. Since the sequence $\langle(\xi,\eta(f))\rangle$ converges in X to $(\omega_0, \eta(f))$, the sequence $\langle(\xi,\omega_1)\rangle$ is fundamental, i.e. for each $f \in C^*(X)$ there exists $\lim_{\xi \to \omega_0} f((\xi,\omega_1)) = \lim f(\xi) = A(f)$. From this it follows easily that the sequence $\langle(\xi,\omega_1)\rangle$ converges to a point in $\beta X - X$. Really, put $Y = X \cup \{(\omega_0,\omega_1)\}$, for each $f \in C^*(X)$ define $f((\omega_0,\omega_1)) = A(f)$, and equip Y with the weak topology with respect to all such extensions. Then X is a dense C^*-embedded subspace of Y. Hence Y is homeomorphic to a subspace of βX and the homeomorphism is pointwise fixed on X.

References

[1] R.Engelking: General topology, Warszawa 1977

[2] R.Frič and V.Koutník: Sequentially complete spaces, Czechoslovak Math.J. (to appear)