WSAA 6

Christoph Bandt

Some combinatorial questions related to measure theory

In: Zdeněk Frolík (ed.): Abstracta. 6th Winter School on Abstract Analysis. Czechoslovak Academy of Sciences, Praha, 1978. pp. 13--15.

Persistent URL: http://dml.cz/dmlcz/701113

Some combinatorial questions related to measure theory
Christoph Bandt
Greifswald, German Democratic Republic

---

1. In this lecture we consider a set X, an algebra $\mathcal{A}$ of subsets of X, a set function $\varphi : \mathcal{B} \longrightarrow [0,\infty[$ defined on a subfamily of $\mathcal{A}$ and (finitely additive) measures on $\mathcal{A}$, which dominate φ or are dominated by φ. We define

$$\alpha(\varphi) = \sup \{ \mu X \mid \mu \text{ measure on } \mathcal{A},\ \mu \le \varphi \text{ on } \mathcal{B} \}$$
$$\beta(\varphi) = \inf \{ \nu X \mid \nu \text{ measure on } \mathcal{A},\ \varphi \le \nu \text{ on } \mathcal{B} \}$$

2. Let k be an integer. A finite sequence $\mathcal{C} = (A_1, \ldots, A_m)$ of elements of $\mathcal{B}$ is called a k-fold covering (exact covering, matching, respectively) of X by elements of $\mathcal{B}$, if the sum $\sum_{i=1}^{m} 1_{A_i}$ of the characteristic functions is greater than (equal to, smaller than) $k \cdot 1_X$ (cf.[2],p.419) We define $s(\mathcal{C},\varphi) = \frac{1}{k} \sum_{i=1}^{m} \varphi A_i$.

Theorem 1 There are measures μ^* and ν^* with

a) $\alpha(\varphi) = \mu^* X = \inf \{ s(\mathcal{C},\varphi) \mid \mathcal{C} \text{ is a multiple covering of } X \}$

b) $\beta(\varphi) = \nu^* X = \sup \{ s(\mathcal{C},\varphi) \mid \mathcal{C} \text{ is a multiple matching of } X \}$

c) If φ is monotone and $\mathcal{B} = \mathcal{A}$ we need only consider exact coverings in a) and b).

This fact is contained in a more special form in [4],[6],[7]. Let us emphasize that it provides a connection between measure theory and combinatorics ($\mathcal{C}$ is just a hypergraph).

3. Theorem 1 yields a weak generalization of the well-known maximal network flow theorem of Ford and Fulkerson. Given a network $G=(V,E)$ with source q, sink s and capacity function $c : E \longrightarrow [0,\infty[$ let X denote the set of all elementary paths [2] from q to s. To an edge e corresponds the set $B_e = \{ w \in X \mid e \text{ is in } w \}$. Let $\mathcal{B} = \{ B_e \mid e \in E \}$ and $\varphi B_e = c(e)$. Flows in the graph-theoretic sense correspond to measures on $\mathcal{P}(X)$ dominated by φ on $\mathcal{B}$. By theorem 1, the maximal flow in G is the infimum of capacities of "multiple cuts" (k-fold cut = sequence of edges which meets every path in X at least k times). This is true for multicommodity networks and infinite networks, too.

Ford-Fulkerson's theorem on "simple cuts" follows from the special structure of usual networks: every k-fold cut splits into k disjoint simple cuts.

4. For every φ, $\alpha(\varphi)=\alpha(\eta)$ and $\beta(\varphi)=\beta(\psi)$, where η and ψ are the outer and inner measure on $\mathcal{A}$ generated by φ. Thus, in the following η denotes a submeasure and ψ a superadditive set function which are normalized: $\eta X=\psi X=1$. On infinite algebras $\mathcal{A}$ there exist ψ with $\beta(\psi)=\infty$, and non-trivial examples of η with $\alpha(\eta)=0$ (so-called pathological submeasures) were given by Popov [6], Herer and Christensen [3] and Topsøe [7]. For $X=X_n=\{1,\ldots,n\}$ and $\mathcal{A}=\mathcal{P}(X_n)$, however, $\alpha(\eta)\geq\frac{1}{n}$ and $\beta(\psi)\leq n$ clearly holds. Hence, the following numbers seem to be of interest.

$$\alpha_n = \inf\{\alpha(\eta) \mid \eta \text{ normalized submeasure on } \mathcal{P}(X_n)\}$$

$$\beta_n = \sup\{\beta(\psi) \mid \psi \text{ normalized and superadditive on } \mathcal{P}(X_n)\}$$

At the 3rd Winter School in Stefanova, 1975, Vašak and Preiss discussed the numbers α_n and raised the question: Which is the first number n with $\alpha_n\neq\frac{n}{2(n-1)}$? We think it is eleven but can only prove it lies between 6 and 11. Asymptotic behavior of α_n is easier determined.

<u>Theorem 2</u> a) $1\leq\underline{\lim}\ \alpha_n\cdot\log n\leq\overline{\lim}\ \alpha_n\cdot\log n\leq 2\cdot\log 2$

b) $\lim\ \beta_n:\sqrt{n}=1$

5. The proof of theorem 2a in [1] uses the fact that for a submeasure η on $\mathcal{P}(X_n)$ with small $\alpha(\eta)$ there exist large sets with small η-values and small sets with large η-values. This fact also implies for an arbitrary η:

<u>Theorem 3</u>

$$\beta(\eta)\geq\alpha(\eta)\cdot\exp\left(\frac{1}{\alpha(\eta)}-2\right) \quad \text{for } \alpha(\eta)\neq 0$$

$$\beta(\eta)=\infty \quad \text{for } \alpha(\eta)=0$$

The last assertion may be considered as a contribution to the well-known question of Maharam, wether for every continuous submeasure on a σ-algebra of sets there is an σ-additive measure with the same zero-sets. By theorem 2 of [3] and theorem 4 of [6], this question is equivalent to the problem, wether all pathological submeasures are discontinous. This concerns sequential continuity with respect to order-convergence, but it suffices to show that pathological submeasures are not exhaustive, that means, there is a disjoint sequence $(A_i)_{i=1,2,..}$ of elements of with $\eta A_i \geq \varepsilon$ for all i and a certain positive number ε. The above assertion is much weaker, of course. It only implies the existence of a disjoint sequence (A_i) with $\sum \eta A_i = \infty$.

6. Let us present a combinatorial question. A positive answer to that question would imply the statement, that for every pathological submeasure η on $\mathcal{A}$ and every integer n there are n pairwise disjoint sets in $\mathcal{A}$ with η-value $\geq \frac{1}{3}$. (This is near to a positive solution of Maharam's question.)

Let $M_1, M_2, \ldots, M_n$ be subsets of a set X. We assume that the intersection of (d+1) different M_i-s is always empty. Let γ be a subfamily of $\mathcal{P}(X)$ with the following property: if $A \subseteq M_i$ $(1 \leq i \leq n)$ then $A \in \gamma$ or $M_i - A \in \gamma$. Let $q(\gamma)$ be the maximal cardinality of a disjoint family of elements of γ .

Given n and d determine $q = \min \{ q(\gamma) \mid X, M_i \text{ and } \gamma \text{ as above} \}$

q is not greater than $\frac{n}{d}$. Is it equal to $\frac{n}{d}$? (This is true for d=2). Does there exist a positive number δ with $q \geq \delta \cdot \frac{n}{d}$?

References

[1] C. Bandt On the permeability of submeasures on finite algebras to appear in Coll. Math. 40, No.2

[2] C. Berge Graphs and Hypergraphs North-Holland 1973

[3] W. Herer and J.P.R. Christensen On the existence of pathological submeasures and the construction of exotic topological groups Math. Annalen 213 (1975), p.203-210

[4] P.J. Huber Kapazitäten statt Wahrscheinlichkeiten? Gedanken zur Grundlegung der Statistik Jber.Dtsch.Math.Ver. 78.2 (1976), p.81-92

[5] I.L. Kelley Measures on Boolean algebras Pac.J.Math. 9 (1959), p.1165-1177

[6] Additive and semiadditive functions on Boolean algebras V.A. Popov Sibirskij Math.J. 17 (1976),p.331-339 (in Russian)

[7] F. Topsøe Some remarks concerning pathological submeasures Math. Scand. 38 (1976), p.159-166