FIFTH WINTER SCHOOL (1977)

ON GROTHENDIECK SPACES OF TYPE C(K)

By

Jürgen FLACHSMEYER

In his fundamental work on weakly compact operators Grothendieck presented the following theorem (see Canad. J. Math. 5 (1953)): For a Banach space E the following properties are equivalent:

(1) Every continuous linear map $u: E \longrightarrow Y$ from E into some separable Banach space is weakly compact, i.e. u transmits t unit ball into some relative weakly compact set.

(2) Weak-$*$-convergence and weak convergence for sequences coinc in the dual E'.

Banach spaces with these properties (1) and (2) are called Grothendieck spaces (see for example J. Diestel: Grothendieck spac and vector measures. In: Vector and operator valued measures and applications. Ed. by D.H. Tucker, H.B. Maynard, Acad. Press. Inc. 1973, 97-108.)

The following problem (problem 3 in Diestel's paper is) unsolved: Characterize those compact Hausdorff spaces K for which the Banach space C(K) of all continuous real-valued functions on K is a Grothendieck space.

What is known about this problem?

Let be K a compact Hausdorff space. We will write $K \in G$ iff C(K) is Grothendieck.

Grothendieck (1953): (i) K Stonian (=extremally disconnected) $\Longrightarrow$

Ando (1961): (ii) K σ-extremally disconnected $\Longrightarrow K \in G$.

Semadeni (1964): by another approach received (ii).

Seever (1968): (iii) K an F-space $\Longrightarrow K \in G$.

H. Schaefer (1971) also proved (ii). Of course, (iii)$\Longrightarrow$(ii)$\Longrightarrow$(i).

By the Riesz representation theorem the dual $C'(K)$ can be identifi with the space M(K) of bounded signed Radon measures on K.

Using this approach $K \in G$ gets equivalent to the following.
For every sequence (μ_n) of bounded Radon measures holds:
$\mu_n(f) \to 0 \ \forall f \in C(K) \Rightarrow \mu_n(g) \to 0 \ \forall$ bounded Borel functions g.
Thus a necessary condition for $K \in G$ is that K must be sequentially discrete.
The lecture now explains the following result:
For every infinite compact F-space K the Alexandrov-double K@K is never an F-space.
For every $K \in G$ the Alexandrov-double K@K belongs to G. Thus a good deal of non-F-spaces are in G.
(Remark: The extension of the class G in a suitable way to non-compact spaces was treated in a thesis (Greifswald 1976) by Nguyen Doan Tien).