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## FIFTH WINTER SCHOOL (1977)

ON GROTHENDIECK SPACES OF TYPE  $C(K)$ 

By

Jürgen FLACHSMEYER

In his fundamental work on weakly compact operators Grothendieck presented the following theorem (see *Canad. J. Math.* 5 (1953)):

For a Banach space  $E$  the following properties are equivalent:

- (1) Every continuous linear map  $u: E \rightarrow Y$  from  $E$  into some separable Banach space is weakly compact, i.e.  $u$  transmits the unit ball into some relative weakly compact set.
- (2) Weak- $*$ -convergence and weak convergence for sequences coincide in the dual  $E'$ .

Banach spaces with these properties (1) and (2) are called Grothendieck spaces (see for example J. Diestel: *Grothendieck spaces and vector measures*. In: *Vector and operator valued measures and applications*. Ed. by D.H. Tucker, H.B. Maynard, Acad. Press. Inc. 1973, 97-108.)

The following problem (problem 3 in Diestel's paper is) unsolved: Characterize those compact Hausdorff spaces  $K$  for which the Banach space  $C(K)$  of all continuous real-valued functions on  $K$  is a Grothendieck space.

What is known about this problem?

Let be  $K$  a compact Hausdorff space. We will write  $K \in G$  iff  $C(K)$  is Grothendieck.

Grothendieck (1953): (i)  $K$  Stonian (=extremally disconnected)  $\Rightarrow$

Ando (1961): (ii)  $K$   $\mathcal{C}$ -extremally disconnected  $\Rightarrow K \in G$ .

Semadeni (1964): by another approach received (ii).

Seever (1968): (iii)  $K$  an  $F$ -space  $\Rightarrow K \in G$ .

H. Schaefer (1971) also proved (ii). Of course, (iii)  $\Rightarrow$  (ii)  $\Rightarrow$  (i).

By the Riesz representation theorem the dual  $C'(K)$  can be identified with the space  $M(K)$  of bounded signed Radon measures on  $K$ .

Using this approach  $K \in G$  gets equivalent to the following.

For every sequence  $(\mu_n)$  of bounded Radon measures holds:

$$\mu_n(f) \rightarrow 0 \quad \forall f \in C(K) \Rightarrow \mu_n(g) \rightarrow 0 \quad \forall \text{ bounded Borel functions } g.$$

Thus, a necessary condition for  $K \in G$  is that  $K$  must be sequentially discrete.

The lecture now explains the following result:

For every infinite compact  $F$ -space  $K$  the Alexandrov-double  $K \circledast K$  is never an  $F$ -space.

For every  $K \in G$  the Alexandrov-double  $K \circledast K$  belongs to  $G$ . Thus a good deal of non- $F$ -spaces are in  $G$ .

(Remark: The extension of the class  $G$  in a suitable way to non-compact spaces was treated in a thesis (Greifswald 1976) by Nguyen Doan Tien).