WSAA 4

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In: Zdeněk Frolík (ed.): Abstracta. 4th Winter School on Abstract Analysis. Czechoslovak Academy of Sciences, Praha, 1976. pp. 135--139.

Persistent URL: http://dml.cz/dmlcz/701062

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FOURTH WINTER SCHOOL (1976)

POSITIVE DEFINITE FUNCTIONS ON ABELIAN SEMIGROUPS

by

Paul RESSEL

The lecture concerns common work, done in København by Christian BERG, Jens Peter Reus CHRISTENSEN and myself.

Let $(S,+)$ be an abelian semigroup with neutral element 0 .

Def. $f: S \longrightarrow \mathbb{R}$ is positive definite iff f is bounded and

$$\sum_{i,j=1}^{n} \alpha_i \alpha_j f(t_i + t_j) \geq 0 \qquad \begin{array}{l} \forall\ (\alpha_1,\ldots,\alpha_n) \in \mathbb{R}^n \\ \forall\ (t_1,\ldots,t_n) \in S^n \\ \forall\ n \in \mathbb{N} . \end{array}$$

$\varphi : S \to [-1,1]$ is a semicharacter:

$$* \begin{cases} (1) & \varphi(0) = 1 \\ (2) & \varphi(s+t) = \varphi(s)\varphi(t) \quad \forall\ s,t \in S . \end{cases}$$

$\hat{S} := \{\varphi : \varphi$ is semicharacter on $S\} \subseteq [-1,1]^S$ is a compact abelian semigroup in the topology of pointwise convergence.

Example: $S = \mathbb{N}_0 := \{0,1,2,\ldots\}$ with addition.

$$\begin{array}{l} [-1,1] \longrightarrow \hat{\mathbb{N}}_0 \\ a \longmapsto (n \longrightarrow a^n) \end{array}$$ is a topol. semigroup isomorphism.

$\mathcal{P} = \mathcal{P}(S) := \{f : f$ is positive definite on $S\}$
$\mathcal{P}_1 := \{f \in \mathcal{P} : f(0) = 1\}$

Lemma: $f \in \mathcal{P} \succ \sup\limits_{s\ S} |f(s)| = f(0)$. In particular we get that $\mathcal{P}$ is closed and $\mathcal{P}_1$ is compact. Of course $\hat{S} \subseteq \mathcal{P}_1$.

Theorem. $\mathcal{P}_1$ is a Choquet simplex and extr $(\mathcal{P}_1) = \hat{S}$. In particular $\forall f \in \mathcal{P}\ \exists\ !$ Radon measure $\mu \in M_+(\hat{S})$ giving the desintegration

$$f(s) = \int_{\hat{S}} \rho(s)\, d\mu(\rho) \qquad \forall\ s \in S.$$

Def. $\psi : S \longrightarrow [0, \infty[$ is called negative definite iff

$$(\psi(s_i) + \psi(s_j) - \psi(s_i + s_j))_{i,j=1,\ldots,n}$$

is pos. semidef. $\forall\ (s_1, \ldots, s_n) \in S^n, \quad \forall\ n \in \mathbb{N}$.

Proposition. Let $\psi : S \longrightarrow [0, \infty[$. Then the following are equivalent:

(i) $\psi \in \mathcal{N}$

(ii) $e^{-t\psi} \in \mathcal{P} \qquad \forall\ t > 0$

(iii) $\sum\limits_1^m \alpha_i = 0 \succ \sum\limits_{i,j} \alpha_i \alpha_j \psi(s_i + s_j) \leq 0$.

Here $\mathcal{N}$ denotes the cone of all neg. def. functions.

Theorem. Let $\psi \in \mathcal{N}$. Then there are uniquely determined

1) $c \in [0, \infty[$

2) $h: S \longrightarrow [0, \infty[$ additive

3) a non-negative Radon measure μ on $\hat{S} - \{1\}$ such that

$$\psi(s) = c + h(s) + \int_{\hat{S}\setminus\{1\}} (1 - \rho(s))\, d\mu(\rho) \quad \forall s \in S.$$

Here $c = \psi(0)$ and $h(s) = \lim_{n \to \infty} \frac{\psi(ns)}{n}$.

Let $f: S \longrightarrow [0, \infty[$, $a_1, \ldots, a_n \in S$.

$$\nabla_1 f(s; a_1) := f(s) - f(s + a_1)$$

$$\nabla_n f(s; a_1, \ldots, a_n) := \nabla_{n-1} f(s\; a_1, \ldots, a_{n-1}) -$$

$$- \nabla_{n-1}(s + a_n; a_1, \ldots, a_{n-1})$$

Def. (CHOQUET)

f is called monotone of infinite order:

$* \; \nabla_n f(s; a_1, \ldots, a_n) \geq 0$

f is called alternating of infinite order:

$* \; \nabla_n f(s; a_1, \ldots, a_n) \leq 0$

$\forall\, s, a_1, \ldots, a_n \in S$ and $\forall\, n \in \mathbb{N}$.

Theorem. a) $\mathcal{M} \subseteq \mathcal{P}$, $\mathcal{M}$ is an extreme subcone of $\mathcal{P}$.

b) $\mathcal{A} \subseteq \mathcal{N}$, $\mathcal{A}$ " " " " " $\mathcal{N}$.

c) If S is 2-divisible (i.e. $\forall\, s \in S \;\exists\; t \in S: s = 2t$) then

$$\mathcal{M} = \mathcal{P} \quad \text{and} \quad \mathcal{A} = \mathcal{N}.$$

Here $\mathcal{M}(\mathcal{A})$ stands for the cone of monotone (alterating) functions of infinite order.

Theorem. Let $\psi \in \mathcal{N}$ have the representation

$$\psi(s) = c + h(s) + \int_{\hat{S}\setminus\{1\}} (1 - \varphi(s))\, d\mu(\varphi).$$

Then $\psi \in \mathcal{A}$ iff μ is concentrated on $(\hat{S} - \{1\})_+$.

Applications.

1) The classical Laplace-Transformation.

Theorem. $f: \mathbb{R}_+^n \longrightarrow \mathbb{R}$ is Laplace-Transform of a finite non-negative measure on $\mathbb{R}_+^n$ iff f is continuous and positive definite.

2) The semigroup $([0,1], \wedge)$.

Proposition. a) f is positive definite $\Leftrightarrow$ $f \geq 0$ and f is increasing

b) f is negative definite $\Leftrightarrow$ $f \geq 0$ and f is decreasing.

3) The semigroup $(L_1^\infty([0,1]), \cdot)$.

We mean the unit ball in L^∞ with multiplication of equivalence classes and the $\sigma(L^\infty, L^1)$ - topology. It is a compact metrizable space, but the semigroup operation is only separately continuous.

$$\varphi : L_1^\infty([0,1]) \longrightarrow \mathbb{R} \quad , \quad \varphi(f) := \int_0^1 f(t)\, dt$$

is continuous and pos. def., but the unique representing prob. measure on $\widehat{L_1^\infty}$ can be shown to be concentrated on a compact subset of the semicharacters, none of which is continuous in the neutral element of L_1^{∞} .

Open Problem: Is this pathology impossible, if the semigroup is for ex. compact (or locally compact) and the addition is jointly continuous ?