I. Namioka
Asplund spaces

In: Zdeněk Frolík (ed.): Abstracta. 4th Winter School on Abstract Analysis. Czechoslovak Academy of Sciences, Praha, 1976. pp. 125--128.

Persistent URL: http://dml.cz/dmlcz/701061

FOURTH WINTER SCHOOL (1976)

ASPLUND SPACES

by

I. NAMIOKA

I reported an instance of two separate lines of investigations that were eventually joined profitably. Much of the material was taken from the joint work with R.R. Phelps.

I. Def. A function f of a topological space X into Y is called barely continuous (by E. Michael) if for each closed subset A of X, the restriction $f|_A$ is continuous at at least one point of A.

Theorem. Let (E,J) be a metrizable locally convex space, and let A be a weakly compact subset of E. Then the identity map of (A, weak) $\longrightarrow$ (A,J) is barely continuous.

The analogue of this theorem is false if "weak" is replaced by "weak*" in a general dual Banach space. So we make the following definition:

Def. The dual E^* of a Banach space E is said to be (DA) if for each weak*-compact subset A of E the identity map (A, weak*) $\longrightarrow$ (A, norm) is barely continuous.

Theorem 1. Suppose that E^* is (DA). Then:

(1) E^* has the Radon-Nikodým Property (RNP).

(2) E^* has the Krein-Milman Property (KMP).

(3) Each w^*-compact convex subset C of E^* is the weak* convex closed hull of those points of C that are

strongly exposed by points of E. (An element f of C is said to be strongly exposed by $x_0 \in E$ if $f(x_0) = \sup\{g(x_0): g \in C\}$ and if, for each net $\{f_\alpha\}$ in C, $f_\alpha(x_0) \longrightarrow f(x_0) \Longrightarrow$ $\Longrightarrow \|f_\alpha - f\| \longrightarrow 0$.)

(Remark. One now knows that for a dual Banach space RNP and KMP are equivalent.)

Examples of E^* that are (DA):

i) Separable E^*

ii) More generally, weakly compactly generated (WCG)E^*.

Problem (Zizler) Is it enough to assume that E^* is contained in some WCG Banach space?

iii) E^* has property (∗∗): A net f_α in E^* converges to f in the norm if $f_\alpha \xrightarrow{w^*} f$ and $\|f_\alpha\| \longrightarrow \|f\|$ (e.g. $\ell^1(\Gamma) = c_0(\Gamma)^*$).

II. A convex function f on R^n can be differentiated a.e. In 1968 Acta Math. paper, Asplund investigated the corresponding situation for convex functions on Banach spaces.

Def. A Banach space E is called an Asplund space (called a strongly differentiability space by Asplund) if each continuous convex function on a convex open subset of E is Fréchet differentiable at each point of a dense subset of the domain.

Asplund proved:

Theorem 2. If E admits an equivalent norm whose dual norm is locally uniformly convex, then E is an Asplund space. (Note: Such a norm has the dual norm that satisfies (∗∗).)

Cor. If E^* is separable, E is Asplund. Also, if E is reflexive E is Asplund.

III. (Synthesis)

Theorem 3. A Banach space E is an Asplund space iff E^* is (DA). (The following result was independently obtained by Collier, John-Zizler, and Namioka-Phelps.)

Cor. If E^* is WCG, then E is an Asplund space (see Example I(ii)).

This new characterization enables us to prove good permanence properties of Asplund space. Asplund proves that if E is Asplund then E/F is Asplund for an arbitrary closed subspace $F \subset E$.

Theorem ① If E is an Asplund space, then each closed subspace is an Asplund space.

② Let E be a Banach space and let F be a closed subspace such that F and E/F are Asplund spaces. Then E is an Asplund space.

③ Let $\{E_\gamma ; \gamma \in \Gamma\}$ be an arbitrary family of Asplund spaces. Then the c_0 and ℓ_p $(1<p<\infty)$ products of $\{E_\gamma\}$ is an Asplund space.

Additional Comments.

i) If E admits an equivalent norm that is Fréchet differentiable (everywhere!), then E is an Asplund space. (Proved by two French mathematicians.)

Problem: Is the converse true ?

ii) For E^*, are (DA) and RNP equivalent? They are kn to be equivalent in the following cases: E is a subspace of a WCG Banach space; $E = C(X)$ for compact Hausdorff X.