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A SURVEY ON LIFTING IN MEASURE THEORY

by

Siegfried GRAF

Def.: For a Boolean algebra $\mathfrak{A}$ and an ideal $\mathcal{N} \subset \mathfrak{A}$ with $1 \notin \mathcal{N}$ a lifting is a map $\varphi : \mathfrak{A} \to \mathfrak{A}$, s.t. (i) $A \sim B \Longrightarrow \varphi(A) = \varphi(B)$, (ii) $A \sim \varphi(A)$, (iii) $\varphi(A \cap B) = \varphi(A) \cap \varphi(B)$, (iv) $\varphi(A \cup B) = \varphi(A) \cup \varphi(B)$, (v) $\varphi(0) = 0$, $\varphi(1) = 1$ for all $A, B \in \mathfrak{A}$. ($A \sim B$ means $A \triangle B \in \mathcal{N}$)

A density is a map $\varphi : \mathfrak{A} \to \mathfrak{A}$ which satisfies (i),(ii), (iii), and (v).

A monotone lifting φ only satisfies (i),(ii), and $A \subset B \Longrightarrow \varphi(A) \subset \varphi(B)$ for all $A, B \in \mathfrak{A}$.

For a set X, a field $\mathfrak{A} \subset \mathfrak{P}(X)$ and an ideal $\mathcal{N} \subset \mathfrak{A}$ with $X \notin \mathcal{N}$ let $\mathcal{L}^\infty(X,\mathfrak{A})$ be the space of bounded, real-valued, $\mathfrak{A}$-meas. functions on X with sup-norm.

A lifting for $\mathcal{L}^\infty(X,\mathfrak{A})$ w.r.t. $\mathcal{N}$ is a map $\ell : \mathcal{L}^\infty(X,\mathfrak{A}) \to \mathcal{L}^\infty(X,\mathfrak{A})$ s.t. (i) $f \sim g \Longrightarrow \ell(f) = \ell(g)$, (ii) $\ell(f) \sim f$, (iii) ℓ linear, (iv) ℓ multiplicative, (v) $\forall \alpha \in \mathbb{R} : \ell(\alpha) = \alpha$

A linear lifting only satisfies (i), (ii), and (iii), while a monotone linear lifting is monotone and satisfies (v) in addition.

For a linear lifting $\ell : \mathcal{L}^\infty(X,\mathfrak{A}) \to \mathcal{L}^\infty(X,\mathfrak{A})$ we define $\|\ell\| = \sup\{\|\ell(f)\| : \|f\|'_{\mathcal{N}} \leq 1\}$, where $\|\ell(f)\|$ denotes the sup-norm of $\ell(f)$ and $\|f\|_{\mathcal{N}} := \inf\{\alpha \in \mathbb{R} : \{|f| \geq \alpha\} \in \mathcal{N}\}$.

Prop. 1: (Ionescu-Tulcea 1965) $\mathcal{A}$ σ-field, $\mathcal{N}$ σ-ideal. If L is a lifting for $\mathcal{A}$ w.r.t. $\mathcal{N}$ then there exists exactly one lifting ℓ for $\mathcal{L}^{\infty}(X,\mathcal{A})$, s.t. $1_{L(A)} = \ell(1_A)$ $\forall A \in \mathcal{A}$, and vice versa.

I. On the existence of densities and liftings

Let $\mathcal{A}$ be a Boolean algebra and $\mathcal{N} \subset \mathcal{A}$ an ideal.

Theorem 1: (v. Neumann-Stone 1935)
If $\mathcal{N}$ is k-complete for all $k <$ card $(\mathcal{A}|\mathcal{N})$, then there is a lifting for $\mathcal{A}$ w.r.t. $\mathcal{N}$.

Theorem 2: (v. Neumann-Stone 1935)
If there is a density for $\mathcal{A}$ w.r.t. $\mathcal{N}$, and if $\mathcal{N}$ is conditionally-k-complete for all $k <$ card $(\mathcal{A}|\mathcal{N})$, then there is a lifting for $\mathcal{A}$ w.r.t. $\mathcal{N}$.

Remark: (v. Weizsäcker 1975)
There exists a Boolean algebra $\mathcal{A}$ and an ideal $\mathcal{N} \subset \mathcal{A}$, s.t. there is a density φ for $\mathcal{A}$ w.r.t. $\mathcal{N}$ but no lifting ψ with $\varphi(A) \subset \psi(A)$ $\forall A \in \mathcal{A}$.

Problem: Is there a lifting for $\mathcal{A}$ w.r.t. $\mathcal{N}$, if there is a density for $\mathcal{A}$ w.r.t. $\mathcal{N}$?

Corollary 1: (Gapaillard 1972)
If $\mathcal{A} \subset \mathcal{P}(X)$ is a field and $\mathcal{N} \subset \mathcal{A}$ an ideal in $\mathcal{P}(X)$ with $X \notin \mathcal{N}$, s.t. there exists a density for $\mathcal{A}$ w.r.t. $\mathcal{N}$, then there is a lifting for $\mathcal{A}$ w.r.t. $\mathcal{N}$.

Corollary 2: (Graf 1972)

If X is a top. space, $\mathfrak{A}$ the σ-field of all sets with Baire property in X and $\mathcal{N}$ the σ-ideal of all sets of first category, then there is a lifting for $\mathfrak{A}$ w.r.t $\mathcal{N}$.

Theorem 3: (v. Weizsäcker-Graf 1973)

If $(X, \mathfrak{A}, \mu)$ is a σ-finite measure space and $\mathcal{N} = \{A \in \mathfrak{A} : : \mu(A) = 0\}$, then there exists a lower density for $\mathfrak{A}$ w.r.t. $\mathcal{N}$.

Corollary: (von Neumann 1931, D. Maharam 1958)

If $(X, \mathfrak{A}, \mu)$ is complete in addition, then there is a lifting for $\mathfrak{A}$ w.r.t. $\mathcal{N}$.

Remarks: The question of existence of a lifting for the unit interval with Lebesgue measure was raised by Haar and positively answered by von Neumann in 1931. In 1958 D. Maharam generalized the theorem to arbitrary σ-finite measure spaces. In 1974 Erdös showed that there is a finitely additive measure on $\mathcal{P}(\mathbb{N})$, s.t. there is no lifting for $\mathcal{P}(\mathbb{N})$ w.r.t. $\{\mu = 0\}$.

Problems:

Which pairs $(\mathfrak{A}, \mu)$ admit a lifting (density)?

To be more specific: Let $\mathfrak{A}$ be a Boolean σ-algebra, $\mathcal{N} \subset \mathfrak{A}$ a σ-ideal, s.t. $\mathfrak{A} | \mathcal{N}$ is weakly countably distributive and satisfies the countable chain condition. Does a density (lifting) for $\mathfrak{A}$ w.r.t. $\mathcal{N}$ exist?

Does every σ-finite measure space admit a lifting?

II. Conditions for a complete measure space, which are equivalent to the existence of a lifting

Let $(X, \mathcal{A}, \mu)$ be a complete measure space, s.t. $\mathcal{A} = \{A \subset X : \forall E \in \mathcal{A} : \mu(E) < \infty \Rightarrow E \cap A \in \mathcal{A}\}$ and define $\mathcal{N} := \{A \in \mathcal{A} : \forall E \in \mathcal{A} : \mu(E) < \infty \Rightarrow \mu(E \cap A) = 0\}$.

$(X, \mathcal{A}, \mu)$ has the lifting property (LP), monotone lifting property (MLP) or the density property (DP), iff there is a lifting, monotone lifting, density for $\mathcal{A}$ w.r.t. $\mathcal{N}$.

a) Decomposition of a measure space

Def.: $\mathcal{Z} \subset \mathcal{A}$ is called a decomposition, iff (i) $\forall Z, Z' \in \mathcal{Z} : Z \neq Z' \Rightarrow Z \cap Z' = \emptyset$, (ii) $\forall Z \in \mathcal{Z} : 0 < \mu(Z) < \infty$, and (iii) $\forall A \in \mathcal{A}$ $(\mu(A) < \infty$ and $\forall Z \in \mathcal{Z} : \mu(A \cap Z) = 0) \Rightarrow \mu(A) = 0$.

Remark: Every Radon-measure-space has a decomposition (of compact sets).

Theorem: The following are equivalent (T.F.A.E.):

(a) $(X, \mathcal{A}, \mu)$ has LP; (b) $(X, \mathcal{A}, \mu)$ has DP; (c) $(X, \mathcal{A}, \mu)$ has MLP; (d) there is a decomposition for $(X, \mathcal{A}, \mu)$;

(e) $\mathcal{L}^\infty(X, \mathcal{A})$ has a linear lifting ℓ with $\|\ell\| < 2$

(f) $\mathcal{L}^\infty(X, \mathcal{A})$ has a monotone linear lifting.

Remark:

(a) $\Longleftrightarrow$ (b) $\Longleftrightarrow$ (d) $\Longleftrightarrow$ (f) was proved by Ionescu-Tulcea and Kölzow 1968.

(d) $\Longleftrightarrow$ (e) was proved by Strauss 1974

(c) $\Longleftrightarrow$ (d) was proved by Gapaillard in 1971.

b) Radon-Nikodym theorem

Def.: $(X, \mathcal{A}, \mu)$ has the Radon-Nikodym property (RNP), iff for every measure ν on $\mathcal{A}$, s.t. $\nu(N) = 0$ for all $N \in \mathcal{A}$ with $\mu(N) = 0$ (i.e. ν is μ-continuous), there is an

$\mathcal{A}$-measurable $f: X \to [0,+\infty]$, s.t. $\forall A \in \mathcal{A}$: $\mu(A) < \infty \Rightarrow (\int_A f d\mu < \infty \Leftrightarrow \nu(A) < \infty)$ and $\nu(A) = \int_A f d\mu$.
f is called a derivative of ν w.r.t. μ.

Remark: (Segal 1951)

$(X, \mathcal{A}, \mu)$ RNP $\Leftrightarrow$ $\mathcal{A} | \mu$ complete lattice $\Leftrightarrow$ $L^{\infty}(X, \mathcal{A}, \mu)$ cond. complete lattice.

Prop. 1: (Kölzow 1968)

$$(X, \mathcal{A}, \mu) \text{ IP} \Rightarrow (X, \mathcal{A}, \mu) \text{ RNP}$$

Remark: (Fremlin 1973)

The converse of the above theorem does not hold.

Def.: $(X, \mathcal{A}, \mu)$ has the monotone (linear) RNP, iff for every measure ν on $\mathcal{A}$ with $\nu \leq \alpha\mu$ for some $\alpha \in \mathbb{R}_+$ there is a derivative f_ν, s.t. for any two of those measures ν_1, ν_2 we have $\nu_1 \leq \nu_2 \Rightarrow f_{\nu_1} \leq f_{\nu_2}$
$(f_{(\alpha_1\nu_1 + \alpha_2\nu_2)} = \alpha_1 f_{\nu_1} + \alpha_2 f_{\nu_2}$ $\forall \alpha_1, \alpha_2 \in \mathbb{R}_+$).

Theorem: (Kölzow 1968)

T.F.A.E.: (i) $(X, \mathcal{A}, \mu)$ has IP; (ii) $(X, \mathcal{A}, \mu)$ has the monotone RNP; (iii) $(X, \mathcal{A}, \mu)$ has the linear RNP.

c) Riesz theorem

Def.: $(X, \mathcal{A}, \mu)$ has the Riesz property (RP), iff $\forall \varphi \in (L^1(X, \mathcal{A}, \mu))' \exists f_\varphi \in \mathcal{L}^{\infty}(X, \mathcal{A}, \mu)$, s.t. $\forall f \in \mathcal{L}^1(X, \mathcal{A}, \mu)$: $\varphi(f) = \int f f_\varphi \, d\mu$.
The map $\varphi \mapsto f_\varphi$ is called an R-differentiation.
$(X, \mathcal{A}, \mu)$ has the monotone (linear) RP, iff there is a monotone (linear) R-differentiation.

Remark: (Segal 1951)

$(X, \mathcal{A}, \mu)$ has RP $\Leftrightarrow$ $(X, \mathcal{A}, \mu)$ has RNP

Theorem: (Kölzow 1968)

T.F.A.E.: (i) $(X,\mathfrak{A},\mu)$ has LP; (ii) $(X,\mathfrak{A},\mu)$ has the monotone RP; (iii) $(X,\mathfrak{A},\mu)$ has the linear RP.

d) Dunford-Pettis theorem

Def.: $(X,\mathfrak{A},\mu)$ has the Dunford-Pettis property (DPP), iff for all Banach spaces B and all bounded linear maps $\mathcal{U}$: $L^1(X,\mathfrak{A},\mu) \longrightarrow B'$ there is a weak*-measurable $f_{\mathcal{U}} : X \longrightarrow B'$, s.t.

$$\forall f \in L^1(X,\mathfrak{A},\mu)\ \forall b \in B:\ [\mathcal{U}(f)](b) =$$
$$= \int f(x)\,[f_{\mathcal{U}}(x)](b)\,d\mu(x).$$

The map $\mathcal{U} \mapsto f_{\mathcal{U}}$ is called a DP-differentiation.

$(X,\mathfrak{A},\mu)$ has the linear (isometric) DPP, iff there is always a linear (isometric) DP-differentiation. Here $\mathcal{U} \mapsto f_{\mathcal{U}}$ is called isometric, iff

$$\|\mathcal{U}\| = \inf\{\alpha \in \mathbb{R} : \{x \in X : \|f_{\mathcal{U}}(x)\| \geq \alpha\} \in \mathcal{N}\}$$

Theorem: (Dieudonné 1951, Ionescu-Tulcea 1962)

T.F.A.E.: (i) $(X,\mathfrak{A},\mu)$ has the LP; (ii) $(X,\mathfrak{A},\mu)$ has the linear DPP; (iii) $(X,\mathfrak{A},\mu)$ has the isometric DPP.

e) Vitali differentiation systems

Def.: For $x \in X$ let $\overline{\mathfrak{A}}(x) \subset \{\mathcal{G} \mid \mathcal{G} \subset \{A \in \mathfrak{A} : 0 < \mu(A) < \infty\}\}$, s.t.: $\forall \mathcal{G}' \subset \{A \in \mathfrak{A} : 0 < \mu(A) < \infty\}$, s.t. $\exists \mathcal{G} \in \overline{\mathfrak{A}}(x) : \mathcal{G}' \subset \mathcal{G}$ and $\forall G \in \mathcal{G}\ \exists G' \in \mathcal{G}' : G' \subset G$, then $\mathcal{G}' \in \overline{\mathfrak{A}}(x)$. In this case $\overline{\mathfrak{A}}(x)$ is called a differentiation system for x.

$\overline{\mathfrak{A}} = (\overline{\mathfrak{A}}(x))_{x \in D}$ is called a differentiation system for $(X,\mathfrak{A},\mu)$, iff $D \in \mathfrak{A}$, $X \setminus D \in \mathcal{N}$ and $\overline{\mathfrak{A}}(x)$ is a diff. system for x for all $x \in D$.

For $B \subset X$, $\mathcal{V} \subset \mathfrak{A}$ is an $\overline{\mathfrak{A}}$-cover for B, iff there is an

$N \in \mathcal{N}$, s.t. $B \setminus N \subset D$ and $\forall x \in B \setminus N \; \exists \mathcal{G} \in \overline{\mathcal{U}}(x)$ with C_x ...

$\mathcal{V} \subset \mathcal{A}$ is a strong (weak) Vitali cover for B, iff $\forall \varepsilon > 0 \; \forall C \subset B$: $0 < \mu^*(C) < \infty \Longrightarrow \exists \; (V_n)_n \subset \mathcal{V}$, s.t. $V_n \cap V_m = \emptyset \; (n \neq m)$, $\mu^*(C \setminus \bigcup_n V_n) = 0$ and $\mu^*(\bigcup_n V_n \setminus C) < \varepsilon$ (resp. $\mu^*(C \setminus \bigcup_n V_n) = 0$ and $(\sum_n \mu(V_n)) - \mu(\bigcup_n V_n) < \varepsilon$).

A differentiation system $\overline{\mathcal{U}} = (\overline{\mathcal{U}}(x))_{x \in D}$ for $(X, \mathcal{A}, \mu)$ is called strong (weak) Vitali system iff for every $B \subset X$ and for every $\overline{\mathcal{U}}$-cover $\mathcal{V}$ for B, $\mathcal{V}$ is a strong (weak) Vitali cover.

Theorem: (Kölzow 1968)

T.F.A.E.: (i) $(X, \mathcal{A}, \mu)$ has the LP; (ii) $\exists$ strong Vitali system for $(X, \mathcal{A}, \mu)$; (iii) $\exists$ weak Vitali system for $(X, \mathcal{A}, \mu)$.

Remark:

Applications of Vitali systems to differentiation of semigroup-valued measures and integral representations of operators can be found in Sion: A theory of semigroup valued measures, Lecture Notes 355(1974).

f) Lifting topologies and category measure

Prop.: (Gapaillard 1972)

Let m be a monotone lifting for $\mathcal{L}^\infty$, $(f_i)_{i \in I}$ a filtering increasing family in $\mathcal{L}^\infty$, s.t. $f_i \leq m\,(f_i) \leq g \in \mathcal{L}^\infty$. Then $\sup f_i \in \mathcal{L}^\infty$.

Corollary: (Maharam 1958)

If D is a lower density for $\mathcal{A}$ w.r.t. $\mathcal{N}$, then $\bigcup_{i \in I} A_i \in \mathcal{A}$

for any family $(A_i)_{i\in I}$ in $\mathfrak{A}$ with $A_i \subset D(A_i)$ for all $i \in I$.

Def.: For a density D let $\tau_D: \{A \in \mathfrak{A} \mid A \subset D(A)\}$.

Theorem: (A. Ionescu- Tulcea 1967)

τ_D is a topology on X, s.t.

(i) $\tau_D \cap \mathfrak{N} = \{\emptyset\}$

(ii) $\forall A \in \mathfrak{A}\ \exists\ \mathcal{U} \in \tau_D: A \Delta \mathcal{U} \in \mathfrak{N}$

(iii) $K \subset X$ is of first category $\Longleftrightarrow$ K closed and nowhere dense $\Longleftrightarrow$ $K \in \mathfrak{N}$

Def.: $(X, \mathfrak{A}, \mu)$ is called a category measure space iff there exists a topology $\mathcal{T}$ on X, s.t. $\mathfrak{A} = \{$sets with Baire property w.r.t. $\mathcal{T}\}$ and $\mathfrak{N} = \{$sets of first category w.r.t. $\mathcal{T}\}$.

Prop.: (Graf 1973)

T.F.A.E.: (i) $(X, \mathfrak{A}, \mu)$ has the LP; (ii) $(X, \mathfrak{A}, \mu)$ is a category measure space; (iii) there exists a topology on X, which satisfies (i) & (ii) of the above proposition.

III. Further applications of liftings

a) Disintegration of measures

Let S be a top. space, $\mathfrak{B}(S)$ the Borel field of S and $(X, \mathfrak{A})$ a measurable space.

Theorem: (Valadier 1974, Maharam 1973, Saint-Pierre 1975 et al.)

Let $\lambda : \mathfrak{A} \otimes \mathfrak{B}(S) \longrightarrow [0, \infty]$ be a measure, s.t. $\mu = p_X(\lambda)$ and $\nu = p_S(\lambda)$ have the following properties: (i) $(X, \mathfrak{A}, \mu)$ has the LP; (ii) ν is a Radon meas. Then there is a family $(\nu_x)_{x\in X}$ of Radon measures on S, s.t. $x \mapsto \nu_x(B)$ is $\mathfrak{A}$-measurable for all $B \in \mathfrak{B}(S)$ and moreover $\lambda(A\times B) = \int_A \nu_x\ (B) d\mu(x)$ for all $A \in \mathcal{A}$ with $\mu(A) < \infty$.

Corollary:

Let $\nu : \mathcal{B}(S) \longrightarrow [0,\infty]$ be a Radon measure and $f: S \longrightarrow X$ a $\mathcal{B}(S)$- $\mathcal{A}$ -measurable map, s.t. $(X,\mathcal{A},\mu)$ with $\mu = f(\nu)$ has the LP.

Then there is a family $(\nu_x)_{x\in X}$ of Radon measures on S, s.t. $x \longmapsto \nu_x(B)$ is $\mathcal{A}$-measurable for all $B \in \mathcal{B}(S)$ and moreover $\int_A \nu_x(B) d\mu(x) = \nu(B \cap f^{-1}(A))$ for all $A \in \mathcal{A}$ with $\mu(A) < \infty$.

b) Strassen's theorem

Let B be a Banach space, $p: B \longrightarrow \mathbb{R}$ sublinear. Then p is continuous if and only if $\|p\| = \sup\{|p(b)| : b\in B$ and $\|b\| \leq 1\} < \infty$.

Let ℓ be a lifting for $\mathcal{L}^\infty(X,\mathcal{A},\mu)$.

Theorem: (Strassen 1965, Ionescu-Tulcea 1968)

Let $(p_x)_{x\in X}$ be a family of continuous sublinear functionals on B, s.t.

(a) $\forall\, b\in B$: $x \longmapsto p_x(b)$ is $\mathcal{A}$-measurable

(b) $\int \|p_x\| d\mu(x) < \infty$ and $x \longmapsto \|p_x\|$ in $\mathcal{L}^\infty$.

(c) $\exists\, N \in \mathcal{N}\ \forall\, b\in B\ \forall\, x \in X\setminus N$: $\ell(t \longmapsto p_x(b))(x) \leq p_x(b)$

Define $q: B \longrightarrow \mathbb{R}$ by $q(b) := \int p_x(b) d\mu(x)$.

Then q is a continuous sublinear functional and for $\varphi \in B'$ we have:

$\varphi \leq q \iff \exists$ family $(\lambda_x)_{x\in X}$ in B', s.t. $x \longmapsto \lambda_x(b)$ is $\mathcal{A}$-measurable, $x \longmapsto \|\lambda_x\|$ is in $\mathcal{L}^\infty$ and $\lambda_x \leq p_x$ for all $x \in X$.

c) Endomorphisms of L^∞ induced by point-mappings

Let X, Y be locally compact spaces, μ Radon-measure on X and ν a Radon measure on Y. Define $\mu^\bullet : \mathcal{B}(X) \to [0,\infty]$ by $\mu^\bullet(B) = \sup\{\mu(K): K \subset B\}$ and $\nu^\bullet$ in an analogous way:

Def.: Let $T: L^\infty(X,\mu) \to L^\infty(Y,\nu)$ be a Banach algebra homomorphism. T is called normal iff $T1 = 1$ and $T(\sup\limits_{i\in I} f_i) = \sup\limits_{i\in I} T(f_i)$ for all families $(f_i)_{i\in I}$ in $L^\infty(X,\mu)$.

Theorem: (Ionescu-Tulcea 1965, Vesterstrøm-Vils 1968).
Let $T: L^\infty(X,\mu) \to L^\infty(Y,\nu)$ be a normal Banach algebra homomorphism. Then there is a $u: Y \to X$, s.t.

(i) $\forall f: X \to \mathbb{R}$ with compact support and continuous $f \circ u$ is $\mu^\bullet$-measurable

(ii) $\forall\, N \subset Y$ $\nu^\bullet$-nullset: $u^{-1}(N)$ is a $\mu^\bullet$-nullset

(iii) $\forall \tilde{f} \in L^\infty(X,\mu): T\tilde{f} = \widetilde{f \circ u}$.

IV. Liftings with additional properties

a) Strong liftings

Let $(X,\mathcal{T})$ be a top. space, $\mathcal{O}\!\mathcal{l}$ a σ-field on X with $\mathcal{T} \subset \mathcal{O}\!\mathcal{l}$ and $\mathcal{n} \subset \mathcal{O}\!\mathcal{l}$ an ideal.

Def.: A lifting (lower density) φ for $\mathcal{O}\!\mathcal{l}$ w.r.t. $\mathcal{n}$ is called strong, iff $\mathcal{U} \subset \varphi(\mathcal{U})$ for all $\mathcal{U} \in \mathcal{T}$.

Lemma: (Ionescu-Tulcea)
Let $(X,\mathcal{T})$ be completely regular, L a lifting for $\mathcal{O}\!\mathcal{l}$ w.r.t. $\mathcal{n}$ and ℓ the corresponding lifting for $\mathcal{L}^\infty(X,\mathcal{O}\!\mathcal{l})$. Then:

L is strong if and only if $\forall f \in \mathcal{C}_b(X): \ell(f) = f$.

Theorem: (Graf 1974)

Let μ be a σ-finite measure on $\mathcal{O}\!\mathcal{L}$, s.t. $\mu(\mathcal{U})>0$ $\forall \mathcal{U} \in \mathcal{T}\setminus\{\emptyset\}$, and $(X,\mathcal{T})$ second countable. Then there is a strong lower density for $(X,\mathcal{O}\!\mathcal{L},\mu)$.

Corollary:

If $(X,\mathcal{O}\!\mathcal{L},\mu)$ is complete in addition, then there is a strong lifting for $(X,\mathcal{O}\!\mathcal{L},\mu)$.

Remark: In the case where $(X,\mathcal{T})$ is completely regular or metrizable, the above corollary was proved by several people, for instance by Ionescu-Tulcea, Sion, and Kellerer.

Prop.: (Ionescu-Tulcea 1969)

Let X be a locally compact space and ν a Radon measure on X. There exists a strong lifting for (X,ν) if and only if there is a decomposition $(K_j)_{j\in\mathcal{J}}$ of (X,ν), s.t. K_j is compact, $K_j = \operatorname{supp} \nu_{K_j}$, and (K_j, ν_{K_j}) has a strong lifting.

Corollary: (Ionescu-Tulcea)

If X is a metrizable locally compact space and ν Radon measure on X with supp $\nu = X$, then there is a strong lifting for (X,ν).

Problem: Let X be a locally compact space, ν a Radon measure on X with supp $\nu = X$. Does (X,ν) have a strong lifting?
Due to the above proposition it is enough to solve the problem for compact spaces. Bichteler and C.Ionescu-Tulcea even reduced the problem to products of two-point-spaces and products of unit intervals resp.

Application: Strict disintegration of measures:

Theorem: (Ionescu-Tulcea)

Let X, S be compact and f: S $\longrightarrow$ X continuous, onto.

Furthermore let ν be a Radon measure on S, $\mu = f(\nu)$.
If there is a strong lifting for $(X, \mathcal{B}_\mu(X), \mu)$, then there is a family $(\nu_x)_{x\in X}$ of Radon measures on S, s.t. $x \mapsto \nu_x(B)$ is $\mathcal{B}_\mu(X)$-measurable for all $B \in \mathcal{B}(S)$, supp $\nu_x \subset f^{-1}(x)$ and $\forall A \in \mathcal{B}(X)$: $\nu(B \cap f^{-1}(A)) = \int_A \nu_x(B) d\mu(x)$.

Remark: The above result generalizes to the case where S and X are locally compact and f is Luzin-measurable.

b) Borel liftings

Let X be a top. space and $\mu : \mathcal{B}(X) \longrightarrow [0, \infty]$ a measure.

A lifting for $(X, \mathcal{B}(X), \mu)$ is called a Borel lifting.

Theorem: (v. Neumann-Stone 1935 using continuum hypothesis)
If X is a second countable top. space, then there is a Borel lifting for $(X, \mathcal{B}(X), \mu)$.

Problem:
Does every Radon measure on a compact space have a Borel lifting?
Maher proved that the problem can be reduced to the products of unit intervals.

c) Invariant liftings

Let $(X, \mathcal{O}, \mu)$ be a measure space and S a set of bijective, bimeasurable mappings $g: X \longrightarrow X$ with $g^{-1}(\mu) = \mu$.

Def.: A lifting (density) $\varphi : \mathcal{O} \longrightarrow \mathcal{O}$ is called S-invariant iff $\forall A \in \mathcal{O}\ \forall g \in S$: $\varphi(g(A)) = g(\varphi(A))$.

Theorem: (A. Ionescu-Tulcea)
Let S be an amenable group. Then $(X, \mathcal{O}, \mu)$ has an S-invariant density if and only if $(X, \mathcal{O}, \mu)$ has an S-invariant lifting.

Corollary:

If S is a countable, amenable group, then $(X, \mathfrak{A}, \mu)$ has an S-invariant lifting.

Theorem: (Ionescu-Tulcea 1967)

Let X be a loc. comp.group, μ Haar measure on X, and S the group of left (resp. right) translations on X (i.e. $X \cong S$). Then there exists an S-invariant lifting for (X, μ). Such a lifting is always strong.

Remark: (v. Weizsäcker 1975)

Let (X, μ) be as in the theorem, $S \subsetneqq S'$. Then there is no S'-invariant lifting for (X, μ).

V. Theorems on the non-existence of liftings

Theorem: (von Neumann 1931, Ionescu-Tulcea)

If $(X, \mathfrak{A}, \mu)$ is a measure space, which is not atomic, then there is no monotone linear lifting $\ell : \mathcal{L}_p(X, \mathfrak{A}, \mu) \longrightarrow$ $\longrightarrow \mathcal{L}_p(X, \mathfrak{A}, \mu)$.

Theorem:

If $(X, \mathfrak{A}, \mu)$ is as in the above theorem, then there is no lifting L for $\mathfrak{A}$, s.t. $\forall (A_n)_{n \in \mathbb{N}} \in \mathfrak{A}^{\mathbb{N}} : \bigcap_{n \in \mathbb{N}} L(A_n) =$ $= L(\bigcap_{n \in \mathbb{N}} A_n)$.

VI. Liftings for mappings with values in a top. space

Let $(X, \mathfrak{A}, \mu)$ be a measure space and E a completely regular space.

Def.: $f: X \longrightarrow E$ measurable: $\Longleftrightarrow \forall g \in \mathcal{C}_b(E)$: $g \circ f$ measurable $\mathcal{L}^\infty_E := \mathcal{L}^\infty_E(X, \mathfrak{A}, \mu) := \{ f \in E^X \mid f$ meas., $f(X)$ relatively comp.$\}$

Theorem: (Ionescu-Tulcea 1969)

Let ℓ be a lifting for $\mathcal{L}^\infty(X, \mathcal{A}, \mu)$.

Then there is a uniquely determined map $\ell_E: \mathcal{L}^\infty{}_E \to \mathcal{L}^\infty{}_E$,

s.t.

(i) $\forall g \in \mathcal{C}_b(E)\ \forall f \in \mathcal{L}^\infty_E: g \circ f \sim g \circ \ell_E(f)$

(ii) $(\forall g \in \mathcal{C}_b(E): g \circ f \sim g \circ f') \Longrightarrow \ell_E(f) = \ell_E(f')$
 $\forall f, f' \in \mathcal{L}^\infty_E$

(iii) $\forall f \in \mathcal{L}^\infty{}_E\ \forall g \in \mathcal{C}_b(E): \ell(g \circ f) = g \circ \ell_E(f)$.

Application:

Theorem: (Ionescu-Tulcea 1969)

If $(X_t)_{t \in T} \subset \mathcal{L}^\infty{}_E$ is an E-valued stochastic process on $(X, \mathcal{A}, \mu)$ ($T \subset \overline{\mathbb{R}}$ interval), ℓ, ℓ_E as above. Then $(Y_t)_{t \in T}$ with $Y_t = \ell_E(X_t)$ is a separable modification of $(X_t)_{t \in T}$.

Problems:

Let E be a Banach space (ordered Banach space)

$\mathcal{L}^\infty{}_E(X, \mathcal{A}, \mu)$ = Vector space of Bochner-measurable E-valued functions on X.

Is there a linear (monotone linear) lifting for $\mathcal{L}^\infty{}_E(X, \mathcal{A}, \mu)$?

Can the Banach spaces be characterized, s.t. such a lifting always exists?

What about the analogous questions for bounded weakly measurable E-valued maps?