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In: Zdeněk Frolík (ed.): Abstracta. 4th Winter School on Abstract Analysis. Czechoslovak Academy of Sciences, Praha, 1976. pp. 83--87.

Persistent URL: http://dml.cz/dmlcz/701049

FOURTH WINTER SCHOOL (1976)

ON SUMMATIONS ON LOCALLY COMPACT SPACES

by

Jürgen FLACHSMEYER and Frank TERPE

(The first part was delivered by the first author.)

1. Introductorial remarks:

There is a classical theory of summations (in German: Limitierungstheorie). The classical book on it is that of Zeller-Beekmann: Springer Verlag 1970.

For (real) sequences one can define a limit for usually not convergent sequences, namely by transforming the sequence in a convergent one. For example:

The sequence $x_1 = a$, $x_2 = b$, $x_3 = a$, $x_4 = b, \ldots, x_{2n-1} = a$, $x_{2n} = b, \ldots$ "converges" to

$\frac{a+b}{2}$ after transforming it by the Cesaro-matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & & \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \ldots & & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \ldots & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \frac{1}{n} & \frac{1}{n} & \ldots & & \frac{1}{n} & 0 & \\ \ldots & & & \ldots & & \ldots & \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_n \\ \ldots \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_n \\ \ldots \end{pmatrix}$$

The Cesaro-matrix is a special example of a Toeplitz-matrix, i.e. an infinite real matrix $(a_{ik})_{\substack{i=1,\dots \\ k=1,\dots}}$ for which the rows have absolutely convergent series. The classical theory can be fruitfully generalized to a general summation theory on locally compact spaces.

The report explains some results of the authors, which will appear in 1976 in two papers:

1 Doklady Akad. Nauk SSSR: On a measure theoretic and a topological extension theoretic aspect of summation theory

2 Math. Nachrichten: On summations on locally compact spaces

2. The definition of a summation given by measure families

Let X,T be two not compact but locally compact Hausdorffian spaces. We call the space X the state space, the space T will be called the parameter space.

By a summation S on the state space X with the parameter space T, we mean a family $S = (\mu_t)_{t \in T}$ of bounded Radon measure on X filtered by the filter T which is generated by the complements of compact sets.

Examples of summations:

1) Any sequence $S = (\mu_n)_{n \in \mathbb{N}}$ of bounded Radon measures on an arbitrary state space X is a summation on X with discrete parameter space $\mathbb{N}$.

2) A Toeplitz matrix $A = (a_{ik})_{i,k=1,2,\dots}$ gives a summation with the state space $X = \mathbb{N}$ with parameter space $T = \mathbb{N}$ by the definition $\mu_i(\{k\}) := a_{ik}$,

$$S = (\mu_i)_{i \in \mathbb{N}} .$$

We show that the basic properties of the classical summation given by A are convergence properties of the measure family S.

3) For the connection with stochastic processes (see later) and in general it is very useful to define a summation kernel on X, T as follows:

$\sigma : X \times T \times \mathcal{C}_{00}(x) \longrightarrow \mathbb{R}$ ($\mathcal{C}_{00}(X)$ is the space of continuous functions with compact support)

is a map with the properties:

a) $\sigma(\cdot, t, f) : X \longrightarrow \mathbb{R}$ is for every $t \in T$, $f \in \mathcal{C}_{00}(X)$ a bounded Borel-measurable function on X

b) $\sigma(x, t, \cdot) : \mathcal{C}_{00}(X) \longrightarrow \mathbb{R}$ is for every $x \in X$, $t \in T$ a bounded Radon measure on X.

A summation with the summation kernel σ and an initial measure μ is then given as follows:

$$S(\sigma, \mu) := (\mu_t)_{t \in T} \text{ with } \mu_t(\cdot) = \int_X \sigma(x, t, \cdot)\, \mu(dx)$$

3. Fundamental properties of summations: In analogy to the classical questions are in general basically:

a) S is convergence preserving: this means that every continuous function f on X with a limit at infinity ($f \in \mathcal{C}_a(X)$) has an S-limit too, i.e. $\mu_t(f) \longrightarrow$ in $\mathbb{R}$.

b) S is permanent: this means a), and for every $f \in \mathcal{C}_a(X)$ the limit and the S-limit are the same.

c) S is convergence generating: this means that every cont. bounded function on X has an S-limit

d) S is core-contracting, i.e. for every bounded cont. function f holds

$$[\liminf_{x \to \infty} f(x), \ \limsup_{x \to \infty} f(x)] \supset$$

$$\supset [\liminf_{t \to \infty} Sf(t), \ \limsup_{t \to \infty} Sf(t)].$$

Shortly we say: The core c(f) contains the core c(Sf). These properties are described by the weakly convergence of the measure family $S = (\mu_t)_{t \in T}$ on some compactifications of the state space.

(This second part was delivered by the second author.)

From the above mentioned description of the fundamental properties of summations it is possible to get the fundamental theorems Matrix-summation as corollaries, as there are the Theorem of Kojima-Schur on a Matrix summation to be convergence preserving, the Theorem of Silverman-Toeplitz on a Matrix summation to be permanent, the Theorem of Schur on a Matrix-summation to be convergence generating. An added lemma on the convergence of hyperdiffuse measures on $\beta\mathbb{N}$ makes it then possible to get in a very short manner the famous Theorem of Steinhaus which states that a convergence generating Matrix summation never can be permanent.

If we choose the parameter space $T = R^+$, the positive real line, and if $p : X \times T \times Bo(X) \longrightarrow [0,1]$ is the transition probability of a time homogeneous Markov process [Bo (X) is the Borel field on X], we get a summation kernel by

$$\sigma_p(x,t,f) := \int_X f(y) \cdot p(x,t,dy)$$

for each $f \in \mathcal{C}_{oo}(X)$.

If we take an initial measure μ, we then get the summation $S(\sigma_p, \mu)$ belonging to the transition probability p and the initial measure μ. Then we are in a position to describe properties of the given process by properties of the summation $S(\sigma_p, \cdot)$. For instance we can deduce the existence of an invariant measure μ_o of the given Markov process, if the "Dirac"-summation $S(\sigma_p, \delta_{x_o})$, x_o a certain point of X, has a certain property of weak convergence and the transition probability depends continuously from x.