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CONCRETE REFINEMENTS OF UNIFORM SPACES

by

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Let U be the category of Hausdorff uniform spaces and uniformly continuous mappings. The concrete category $\mathcal{K}$ will be called concrete refinement of U if it has the same objects as U and contains U as a subcategory. The space X will be called $\mathcal{K}$-fine if all $\mathcal{K}$-mappings with domain X are uniformly continuous, $\mathcal{K}$-coarse if all $\mathcal{K}$-mappings with range X are uniformly continuous.

Theorem 1: Let $\mathcal{K}$ be a concrete refinement of U. The class of all $\mathcal{K}$-fine spaces forms a coreflective subcategory of U, the class of all $\mathcal{K}$-coarse spaces forms an epireflective and hereditary subcategory of U. Conversely every coreflective subcategory of U is of the form $\mathcal{K}$-fine and any hereditary epireflective subcategory of U is of the form $\mathcal{L}$-coarse for some concrete refinements $\mathcal{K}$, $\mathcal{L}$.

The concrete refinement $\mathcal{K}$ will be called fine-maximal (resp. coarse-maximal) if it is the largest concrete refinement w.r.t. inclusion generating the same class $\mathcal{K}$-fine (resp. $\mathcal{K}$-coarse).

Theorem 2: The following properties of a concrete refinement are equivalent:

(i) $\mathcal{K}$ is fine-maximal.

(ii) The embedding funçtor U $\hookrightarrow \mathcal{K}$ has an idempotent left adjoint.

(iii) $\mathcal{K}_f X$ is $\mathcal{K}$-isomorphic to X for all spaces X, where $\mathcal{K}_f$ denotes the coreflector into $\mathcal{K}$-fine spaces.

One can also prove the dual theorem for coarse-maximal refinements.

Theorem 3: There are no simultaneously fine-maximal and coarse-maximal concrete refinements of U (except of the trivial refinement U).

Corollary: The only coreflector preserving proximity is the identical functor.

One can find the proofs of these statements in [V]. In [F] one can find interesting examples of refinements.

Example and problem: The mapping between two uniform spaces is called Cauchy if the image of any Cauchy filter is again a Cauchy filter. Obviously Cauchy forms a concrete refinement of U. Furthermore Cauchy is a fine-maximal refinement and Cauchy-fine is a class of all uniform spaces which are dense in topologically fine spaces. It follows from the first statement and Theorem 2 that the functor Cauchy_f preserves the structure of Cauchy filters. There is an interesting question whether there exists some nonidentical modification r (reflection preserving underlying sets) in U preserving Cauchy structure (i.e. rX is Cauchy-isomorphic to X for any space X). This question has several equivalent reformulations :

(i) Is there some nonidentical modification in U commuting with completion ?

(ii) Is there some nonidentical modification such that the value of a complete space is again complete ?

(iii) Is there a coarse-maximal refinement $\mathcal{K}$ (nontrivial) such that $\mathcal{K}$-fine is closed under dense subspaces ? The problem seems to be difficult in general, the best partial result is that such modification must be identical on the class of all distal spaces (spaces having the basis of finite dimensional covers)

References:

[F] Frolík Z.: Basic refinements of the category of uniform spaces TOPO 72 - General Topology and its Applications, Second Pittsburgh International Conference, 1972, 140-158

[P] Pelant J.: Reflections not preserving completeness. Seminar Uniform Spaces 1973-74 directed by Z. Frolík, MÚ ČSAV, Prague.

[V] Vilímovský J.: Categorial refinements and their relation to reflective subcategories. Seminar Uniform Spaces 1973-74 directed by Z. Frolík, MÚ ČSAV, Prague.