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In [S], A.H.Stone raised a question of whether each uniform space has a basis consisting of locally finite covers (recall the A.Stone theorem asserting that each metric space is paracompact).It is shown easily in [I] that the existence of a basis consisting of locally finite covers is equivalent to the existence of a basis consisting of point--finite covers. Stone's problem is restated in [I] and other related problems are pointed out (e.g. the problem of when the Ginsburg--Isbell derivative forms a uniformity, see [P_1],[PPV]). The negative answer to Stone's problem was given independently by E.Ščepin and myself in 1975. Hence the class of all spaces with a point-finite basis forms a "nice" proper epireflective subcategory of UNIF. However, it appears that even spaces having point-finite bases are very wild and that perhaps the best uniform spaces are those having bases consisting of σ-disjoint covers. (A σ-disjoint basis implies the existence of a point-finite base (see e.g. [RR],[P_1], but the converse is not true, (see [P_2])). This paper illustrates the use of "combinatorial" (or discrete) reasoning, as opposed to "continuous" reasoning, in the theory of uniform spaces. This approach seems particularly applicable to problems dealing with covering properties of uniformities.

We are going to estimate point character of some uniform spaces. Finally, we show that the properties of cardinal reflections in UNIF depends on set-theoretical assumptions.

Notation: Let A be a set and let α be a cardinal. We define:

$\mathcal{P}(A) = \{B \mid B \subset A\}$

$[A]^{<\alpha} = \{B \subset A \mid |B| < \alpha\}$, $[A]^{\leq\alpha} = \{B \subset A \mid |B| \leq \alpha\}$

$[A]^{\alpha} = \{B \subset A \mid |B| = \alpha\}$; the meaning of $[A]^{>\alpha}$ and $[A]^{\geq\alpha}$ is obvious.

<u>Definition</u>: Let $\mathcal{A}$ be a collection of sets.

1) The order ($\operatorname{ord}\mathcal{A}$) of $\mathcal{A}$ is defined by $\operatorname{ord}\mathcal{A} = \sup\{|\mathcal{B}| \mid \mathcal{B}\in[\mathcal{A}]^{<\omega_0},\ \bigcap\mathcal{B}\neq\emptyset\}$.

2) The degree ($\deg\mathcal{A}$) of $\mathcal{A}$ is defined by $\deg\mathcal{A} = \max(\omega_0,\ \sup\{|\mathcal{B}|^{+} \mid \mathcal{B}\subset\mathcal{A},\ \bigcap\mathcal{B}\neq\emptyset\}$.

<u>Remark</u>: Following [I], if a uniform space $(X,\mathcal{V})$ has a basis consisting of covers of order at most $(n+1)$, (n is a non-negative integer) and does not have a basis consisting of covers of order at most n, then $(X,\mathcal{V})$ is said to be n-dimensional ($\Delta\delta(X,\mathcal{V}) = n$). If there exists no integer n such that $\Delta\delta(X,\mathcal{V})\leq n$, then we set $\Delta\delta(X,\mathcal{V}) = \infty$.

<u>Definition</u>: The point-character $pc(X,\mathcal{V})$ of a uniform space $(X,\mathcal{V})$ is defined to be the least cardinal α such that $(X,\mathcal{V})$ has a basis consisting of covers whose degrees are at most α.

<u>Basic notation</u>: Let p be a positive integer and let M be a non-empty set. The symbol $\mathcal{K}^p(M)$ denotes the set of all sequences $\{C_j\}_{j=1}^{p}$ such that $C_p\subset M$ and $C_j\subset C_{j+1}$, $j=1,\dots,p-1$. The members of $\mathcal{K}^p(M)$ are called <u>cornets</u> (of length p on a set M).

If $C\in\mathcal{K}^p(M)$, then C_j, $j\in\{1,\dots,p\}$, denotes the j^{th} coordinate of the cornet C, i.e. $C = \{C_j\}_{j=1}^{p}$.

If $V\in\mathcal{K}^{p+1}(M)$, we define $\mathcal{U}(V) = \{C\in\mathcal{K}^p(M) \mid V_j\subset C_j\subset V_{j+1},\ j=1,\dots,p\}$.

Now let $\{D_i\}_{i=1}^{j}$, $j\in\{1,\dots,p\}$ be a sequence of subsets of M. Let $C\in\mathcal{K}^p(M)$. We define $C-\{D_i\}_{i=1}^{j}$ to be the cornet $\widetilde{C}\in\mathcal{K}^p(M)$ satisfying $\widetilde{C}_t = C_t - \bigcup_{i=t}^{j} D_i$, $t=1,\dots,p$.

Let $V\in\mathcal{K}^{p+1}(M)$. We define $V\triangledown\{D_i\}_{i=1}^{j} = \{C\in\mathcal{K}^p(M) \mid C\in\mathcal{U}(V-\{D_i\}_{i=1}^{j})$ and $C_i\cap D_i = \emptyset,\ i=1,\dots,j\}$.

<u>Remarks</u>: The definition of $C-\{D_i\}$ is really correct. $V\triangledown\{D_i\}_{i=1}^{j}\subset\mathcal{U}(V-\{T_i\}_{i=1}^{j+1})$ where $T_i = D_i\cup D_{i-1}$, $i=2,\dots,j$, $T_{j+1} = D_j$, $T_1 = D_1$.

Notation: Let Q be a set, $F \in \mathcal{P}(Q)$, $C \in \mathcal{K}^{p+1}(M)$, $j \in \{1,\dots,p\}$. Let $r : \mathcal{P}(\mathcal{K}^p(M)) \longrightarrow \mathcal{P}(Q)$ be a mapping and let ξ be an infinite regular cardinal less than $|M|$. $A(p,j,F,C)$ denotes the following formula(where the sets X_i and Y_i in $A(p,j,F,C)$ are members of $[M]^{\leq\xi}$) $\forall Y_{j+1} \exists X_j \supset Y_{j+1} \forall Y_j \supset X_j \exists X_{j-1} \supset Y_j \forall Y_{j-1} \supset X_{j-1} \dots \exists X_1 \supset$ $\supset Y_2 \forall Y_1 \supset X_1 : r((C - \{Y_i\}_{i=1}^{j}) \nabla \{X_i\}_{i=1}^{j}) \subset F$.

Basic lemma: Let n be a positive integer. Let M be an uncountable set and let ξ be a regular infinite cardinal less than $|M|$. Let Q be a set. Let $r : \mathcal{P}(\mathcal{K}^n(M)) \longrightarrow \mathcal{P}(Q)$ be a mapping. If the following conditions (0), (1) are satisfied:

(0) for each pair X,Y : if $X \subset Y \subset \mathcal{K}^n(M)$, then $r(X) \subset r(Y)$, and

(1) there exist $j_0 \in \{1,\dots,n\}$ and $C \in \mathcal{K}^{n+1}(M)$ with $|C_1| = |M|$ such that the formula $A(n,j_0,F,C)$ is not valid for any $F \in [Q]^{\leq\xi}$ (i.e. $\exists j_0 \forall C \forall F$: non $A(n,j_0,F,C)$),

then there is $\tilde{C} \in \mathcal{K}^{n+1}(M)$ such that $|r(\mathcal{U}(\tilde{C}))| \geq \xi$.

In addition, we may suppose that $C_i = \tilde{C}_i$ for all $i > j_0$.

Proof: The Basic Lemma can be found in [P3]. We omit the proof due to its length and complexity.

Point-character of uniform box-product

We are going to show that there is a very simple construction of an α-box product which yields uniform spaces of large point-character.

Definition: Given a uniform space $(X, \mathcal{V})$, an infinite cardinal α, and a non-empty index set I, we define a uniform α-box product $\Pi_\alpha X^I = (X^I, \mathcal{V}_\alpha^I)$ as a uniform space whose underlying set is X^I and the basis of the uniformity $\mathcal{V}_\alpha^I$ is formed by all covers of the form:

$$\bigwedge_{s \in S} \pi_s^{-1}(\mathcal{P}) \qquad \text{where } S \in [I]^{<\alpha} \quad \text{and} \quad \mathcal{P} \in \mathcal{V} .$$

Remarks: 0) The uniform α-box product of a zerodimensional uniform space is 0-dimensional.

1) $\Pi_\alpha + R^\alpha$ (where R denotes the uniform space of real numbers) in-

duces the usual uniformity on $l_\infty(\alpha)$, ($l_\infty(\alpha) \subsetneqq R^\alpha$) .

2) One can define the uniform α-box product in a more general setting: it is not necessary to suppose that all coordinate spaces are equal to each other. Even then the following theorem remains valid (the assumption of the following Theorem would then read that at least α coordinate spaces are not 0-dimensional).

<u>Theorem</u>: Let $(X,\mathcal{V})$ be a uniform space that is not 0-dimensional. Let α be an infinite cardinal. If $|I| \geq \alpha$ then $pc(\mathcal{T}_\alpha + X^I) > \xi$ for each regular cardinal $\xi < \alpha$.

<u>Proof</u>: The following lemmas are needed.

<u>Definition</u>: A finite sequence $\{M_i\}_{i=1}^n$ of sets is a chain of length n if: 1) $M_i \cap M_j \neq \emptyset$ iff $|i-j| \leq 1$,
2) $M_{i+1} - \bigcup_{t=1} M_t \neq \emptyset$ for $i=1,\dots,n-1$.

<u>Lemma 1</u>: Let $(X,\mathcal{V})$ be a uniform space. The following conditions are equivalent: 1) $(X,\mathcal{V})$ is not 0-dimensional;
2) there is $\mathcal{O} \in \mathcal{V}$ such that for each $\mathcal{S} \in \mathcal{V}$, there is a chain $\{S_i\}_{i=1}^n$ of members of $\mathcal{S}$ such that $S_1 \cup S_n$ is not contained in any member of $\mathcal{O}$;
3) there is a cover $\mathcal{O} \in \mathcal{V}$ such that there is $\mathcal{P} \in \mathcal{V}$, $\mathcal{P} < \mathcal{O}$ such that for each $\mathcal{R} \in \mathcal{V}$, $\mathcal{R} < \mathcal{P}$ and each $\mathcal{S} \in \mathcal{V}$, $\mathcal{S} < \mathcal{R}$ there is a chain $\{S_i\}_{i=1}^n$ of elements of $\mathcal{S}$ such that $st(S_1,\mathcal{P}) \cap st(S_n,\mathcal{P}) = \emptyset$.

<u>Proof of Lemma 1</u>: (3)$\Rightarrow$(1) is selfevident as non(1)$\Rightarrow$non(3).
(2)$\Rightarrow$(3). Take $\mathcal{P} \in \mathcal{V}$, $\mathcal{P}^{++} \leq \mathcal{O}$ where $\mathcal{O}$ is the cover guaranteed by (2).
(1)$\Rightarrow$(2). We show: non(2)$\Rightarrow$non(1). Suppose that for each $\mathcal{P} \in \mathcal{V}$ there is $\mathcal{K}_\mathcal{P} \in \mathcal{V}$ such that for each chain $\{S_i\}_{i=1}^n$ of elements of $\mathcal{K}_\mathcal{P}$, $S_1 \cup S_n$ is contained in some member of $\mathcal{P}$. Choose $\mathcal{S} \in \mathcal{V}$. We show that there is a uniform refinement of $\mathcal{S}$ of order 1. Choose a

uniform cover $\mathcal{T}$ such that $\mathcal{T} \overset{+}{<} \mathcal{S}$. Consider $\mathcal{L}_{\mathcal{T}}$. We define a relation $\rho \subset X \times X$ by $(x_1, x_2) \in \rho$ iff there is a chain $\{S_i\}_{i=1}^{n}$ of elements of $\mathcal{L}_{\mathcal{T}}$ so that $x_1 \in S_1$, $x_2 \in S_n$. Evidently, ρ is reflexive. Its symmetry and transitivity is given by the following

Lemma 2: Let $\{T_i\}_{i=1}^{n}$ be a system of sets satisfying: $T_i \wedge T_{i+1} \neq \emptyset$, $i=1,\dots,n-1$. Let $x \in T_1$, $y \in T_n$. Then there is a chain $\{S_j\}_{j=1}^{k}$ such that $\{S_j | j=1,\dots,k\} \subset \{T_i | i=1,\dots,n\}$ and $x \in S_1$, $y \in S_k$.

Hence ρ is an equivalence relation that induces a partition $\mathcal{D}$ of X. Evidently, $\mathcal{L}_{\mathcal{T}} < \mathcal{D}$, so $\mathcal{D} \in \mathcal{V}$ and it is easy to check that $\mathcal{D} < \mathcal{S}$ (use $\mathcal{T} \overset{+}{<} \mathcal{S}$). QED.

Lemma 3: Let $(X, \mathcal{V})$ be a uniform space. Let $\mathcal{P} \in \mathcal{V}$ be a cover of $\deg \mathcal{P} = \alpha$. There is $q \in \mathcal{V}$, $q < \mathcal{P}$ and $\mathcal{L} \in \mathcal{V}$ such that each member of $\mathcal{L}$ intersects less then α elements of q.

Proof: Apply the concept of a strict uniform shrinking ([I], Lemma VII. 3).

Proof of Theorem: We proceed by contradiction. Suppose that $(\odot)$: $pc(\pi_\alpha + X^I) \leq \xi$ for some regular cardinal $\xi < \alpha$. Choose $M'' \subset I$, $|M''| = \alpha$. Let $\mathcal{O} \in \mathcal{V}$ and $\mathcal{P} \in \mathcal{V}$ be covers whose existence is given by Lemma 1 (3). Denote $\mathcal{X} = \bigwedge_{m \in M''} \pi_m^{-1}(\mathcal{P})$, hence $\mathcal{X} \in \mathcal{V}_{\alpha^+}^{I}$. By $\odot$ and Lemma 3, there is a uniform cover $\mathcal{Y} = \bigwedge_{m \in M'} \pi_m^{-1}(\mathcal{R})$, $M' \in [I]^{\leq \alpha}$, $\mathcal{R} \in \mathcal{V}$ such that $\mathcal{Y} < \mathcal{X}$ and there is $\mathcal{W}' \in \mathcal{V}_{\alpha^+}^{I}$ such that each $W \in \mathcal{W}'$ intersects less than ξ elements of $\mathcal{Y}$. Using Lemma 1 (3) and properties of a uniformity, we obtain a uniform cover $\mathcal{W} = \bigwedge_{m \in M} \pi_m^{-1}(\mathcal{S})$ $\mathcal{W} \in \mathcal{V}_{\alpha^+}^{I}$, such that $\mathcal{W} < \mathcal{Y}$ and $\mathcal{W} \overset{+}{<} \mathcal{W}'$ and there is a chain $\{S_i\}_{i=1}^{n}$ of elements of $\mathcal{S}$ such that $st(S_1, \mathcal{P}) \cap st(S_n, \mathcal{P}) = \emptyset$ (it follows that the length of this chain is at least 4). Since $\mathcal{W} < \mathcal{Y}$, $M \supset M' \supset M''$, so $|M| = \alpha$. Choose $x_0 \in X$. We define $Z = \{y \in X^I |$ $((m \in I-M) \Longrightarrow (y_m = x_0))$ and $((m \in M) \Longrightarrow (y_m \in \bigcup_{i=1}^{n} S_i))\}$.

Define $\varphi: Z \longrightarrow \mathcal{K}^{n-1}(M)$ by $\varphi(y) = C$ iff $C_j = \{m \in M \mid y_m \in \bigcup_{i=1}^{j} S_i\}$ $j=1,\dots,n-1$.

Observation 1: φ is onto $\mathcal{K}^{n-1}(M)$.

Proof: Use the properties of chains.

Observation 2: Let $V \in \mathcal{K}^n(M)$. For $x,y \in \varphi^{-1}(\mathcal{U}(V))$ and $m \in M$, the following holds: if $x_m \in S_j$ and $y_m \in S_k$ them $|j-k| \leq 1$.

Proof: Suppose $j<k$, hence $j<n-1$. $V_{j+1} \supset \{\iota \in M \mid x_\iota \in \bigcup_{i=1}^{j} S_i\}$, so $m \in V_{j+1}$. But $V_{j+1} \subset \{\iota \in M \mid y_\iota \in \bigcup_{i=1}^{j+1} S_i\}$, hence $y_m \in S_{j+1}$.

Observation 3: For each cornet $V \in \mathcal{K}^n(M)$, the set $\varphi^{-1}(\mathcal{U}(V))$ intersects less than ξ elements of $\mathcal{Y}$.

Proof: Let $x \in \varphi^{-1}(\mathcal{U}(V))$. For each $m \in M$, choose $i(m) \in \{1,\dots,n\}$ so that $x \in \bigcap_{m \in M} \pi_m^{-1}(S_{i(m)})$. Set $\bigcap_{m \in M} \pi_m^{-1}(S_{i(m)}) = W_x$. Evidently, $W_x \in \mathcal{W}$. By Observation 2, $\varphi^{-1}(\mathcal{U}(V)) \subset \mathrm{st}(W_x, \mathcal{W})$. Now use $\mathcal{W} \overset{**}{\leq} \mathcal{W}'$ and the fact that each member of $\mathcal{W}'$ intersects less than ξ elements of $\mathcal{Y}$.

Observation 4: Let $V^1, V^2 \in \mathcal{K}^n(M)$. If $M' \cap (V^1_1 - V^2_n) \neq \emptyset$ (recall that $\mathcal{Y} = \bigwedge_{m \in M'} \pi_m^{-1}(\mathcal{R})$), then there is no $Y \in \mathcal{Y}$ satisfying:

$Y \cap \varphi^{-1}(\mathcal{U}(V^1)) \neq \emptyset \neq Y \cap \varphi^{-1}(\mathcal{U}(V^2))$.

Proof: Put $Y = \bigcap_{m \in M'} \pi_m^{-1}(R(m))$, $R(m) \in \mathcal{R}$ for each $m \in M'$. Let $x \in$ $\in Y \cap \varphi^{-1}(\mathcal{U}(V^1))$. Let $m_0 \in (V^1_1 - V^2_n) \cap M'$. Then $x_{m_0} \in S_1$. Let $y \in Y \cap \varphi^{-1}(\mathcal{U}(V^2))$. Then $y_{m_0} \in S_n - \bigcup_{i=1}^{n-1} S_i \neq \emptyset$. So $R(m_0) \cap S_1 \neq \emptyset \neq$ $\neq R(m_0) \cap S_n - \bigcup_{i=1}^{n-1} S_i$, which contradicts Lemma 1 (3) and $\mathcal{R} < \mathcal{P}$.

Now define a mapping $r: \mathcal{P}(\mathcal{K}^{n-1}(M)) \longrightarrow \mathcal{P}(\mathcal{Y})$ by $r(\mathcal{D}) = \{Y \in \mathcal{Y} \mid$ there is $D \in \mathcal{D}$ such that $\varphi^{-1}(D) \cap Y \neq \emptyset\}$ for $\mathcal{D} \subset \mathcal{K}^{n-1}(M)$.

Observation 5: If $V \in \mathcal{K}^n(M)$ satisfies: $|V_1| = \alpha$, $V_n = M'$ then $A(n,n-1,F,V)$ does not hold for any $F \in [\mathcal{Y}]^{\leq \xi}$.

<u>Proof</u>: Perform an easy prolonged computation using Observation 4. Hence the assumptions of the Basic Lemma are satisfied, so there is $\tilde{V} \in \mathcal{K}^n(M)$ such that $|(r(\mathcal{U}(\tilde{V}))) \geq \xi$, which is a contradiction.

<u>Remarks</u>: 1) The Theorem shows that $pc(\ell_\infty(\alpha)) > \xi$, where ξ is a regular cardinal less than α; in particular, $pc\ \ell_\infty(\omega_1) > \omega_0$. The point character of $\ell_\infty(\omega_0)$ is an open problem, but we feel that it should soon be solved.

2) By similar methods we have partially solved a problem concerning the preservation of Cauchy filters by reflections in Unif (see [P_4], [P_5]). This problem is due to Z.Frolík and particular cases are mentioned in [I],[GI]). Our main result says thet if F is a reflection preserving Cauchy filters, then the spaces in $\{F(\ell_\infty(\alpha)) | \alpha \in Cn\}$ do not have bounded point-character.

<u>Cardinal modifications</u>

<u>Definition</u>: Let ω_α be a cardinal. We define a functor p^α : UNIF $\to$ $\to$ UNIF by $p^\alpha(X, \mathcal{V}) = (X, p^\alpha \mathcal{V})$ where $p^\alpha \mathcal{V}$ consists of all covers $\mathcal{P}_0 \in \mathcal{V}$ such that there is a sequence $\{\mathcal{P}_n\}_{n=1}^\infty \subset \mathcal{V}$ with $|\mathcal{P}_n| < \omega_\alpha$ for $n=1,2,\ldots$ and $\mathcal{P}_n \overset{+}{<} \mathcal{P}_{n-1}$, $n=1,2,\ldots$.

<u>Remark</u>: p^α is a reflection that preserves underlying sets and topology. Such reflections are called modifications.

<u>Definition</u>: Let $(X, \mathcal{V})$ be a uniform space. Let ω_α be a cardinal. We define $b^\alpha \mathcal{V} = \{\mathcal{P} \in \mathcal{V} \mid |\mathcal{P}| < \omega_\alpha\}$.

<u>Remark</u>: Clearly, $(X, b^\alpha \mathcal{V})$ is a quasiuniformity in the sense of [I]. The difficulties are connected with star-refinements. If $(X, b^\alpha \mathcal{V})$ is a uniformity, then $(X, b^\alpha \mathcal{V}) = p^\alpha(X, \mathcal{V})$.

It is well-known that $(X, b^0 \mathcal{V})$ and $(X, b^1 \mathcal{V})$ always form uniformities. A more general theorem, proved in [V] and [K], says: if $pc(X, \mathcal{V}) \leq \omega_0$, then $(X, b^\alpha \mathcal{V})$ is a uniformity for any cardinal ω_α. On the other hand, A.Kucia proved under [GCH] : Let $(X, \mathcal{V})$ be a uniform space. Then $(X, b^{\alpha}\mathcal{V})$ forms a uniformity for any cardinal ω_α. Both these theorems are corollaries of the Folklore Lemma introduced below. Since there are uniform spaces with large point-character it is possible that the equality $p^{\alpha}(X,\mathcal{V}) = (X, b^{\alpha}\mathcal{V})$ depends on set-theoretical assumptions for $\alpha \geq 2$. It is really the case.

Notation: The symbol $S^+(M)$ denotes the positive unit sphere in $\ell_\infty(M)$, i.e. the subspace of $\ell_\infty(M)$ on the set $\{f \in \ell_\infty(M) \mid \|f\| = 1$ and $f(m) \geq 0$ for each $m \in M\}$.

Notation: For each $f \in S^+(M)$ and a non-negative integer k, define $C^{(f,k)} \in \mathcal{K}^{2^k}(M)$ by $C_i^{(f,k)} = f^{-1}(\,]\!]\frac{2^k - i}{2^k}, 1]\!])$, $i = 1, \dots, 2^k$. For $V \in \mathcal{K}^{2^k+1}(M)$, put $\mathcal{W}(V) = \{f \in S^+(M) \mid C^{(f,k)} \in \mathcal{U}(V)\}$.

Definition: For a non-negative integer k, we define $\mathcal{P}_k = \{\mathcal{W}(V) \mid V \in \mathcal{K}^{2^k+1}(M)\}$.

Proposition: $\{\mathcal{P}_k\}_{k=1}^{\infty}$ forms a basis for the norm uniformity on $S^+(M)$.

Proof (see $[P_5]$).

Notation: The mapping which assigns $C^{(f,k)} \in \mathcal{K}^{2^k}(M)$ to each $f \in S^+(M)$ will be denoted by φ_k.

Let ω_α be an uncountable cardinal. $ZB(\omega_\alpha)$ denotes the following assertion: There is $\mathcal{A} \subset [\omega_\alpha]^{\omega_\alpha}$, $|\mathcal{A}| > \omega_\alpha$, an infinite regular cardinal $\xi < \mathrm{cf}\,\omega_\alpha$, and a cardinal $K \leq |\mathcal{A}|$ such that $|\bigcap \mathcal{A}'| < \xi$ for each $\mathcal{A}' \in [\mathcal{A}]^{\geq K}$.

Remark: [GCH] implies that $ZB(\omega_\alpha)$ is false.

Theorem: If $ZB(\omega_\alpha)$ holds, then there is a uniform cover $\mathcal{P}$ of $S^+(\omega_\alpha)$ such that $|\mathcal{P}| = \omega_\alpha$ and $\mathcal{Q} \preceq \mathcal{P}$ implies $|\mathcal{Q}| > \omega_\alpha$ (i.e. $p^{\alpha}S^+(\omega_\alpha) \neq b^{\alpha}S^+(\omega_\alpha)$).

The following lemma is needed.

Folklore Lemma $[P_6]$: Let $(X, \mathcal{V})$ be a uniform space. Let K be an infinite cardinal. Let $\mathcal{P} = \{P_a\}_{a \in A} \in \mathcal{V}$, $|\mathcal{P}| < K$. Let $\mathcal{R} = \{R_b\}_{b \in B}$ be a uniform star-refinement of $\mathcal{P}$. For $x \in X$ put

$S(x) = \{a \mid a \in A$ and $st(x, \mathcal{R}) \subset P_a\}$. For $Y \subset X$, put $I(Y) = \{a \mid a \in A$ and $Y \subset P_a\}$. A mapping $t : X \longrightarrow A$ satisfying $t(x) \in S(x)$ for each $x \in X$ will be called a choice mapping.

Assertion of Folklore Lemma: There is $\mathcal{Q} \in \mathcal{V}$, $\mathcal{Q} = \{Q_a\}_{a \in A}$, such that $\mathcal{Q} \overset{+}{<} \mathcal{P}$ and $\mathcal{R} < \mathcal{Q}$ iff the following condition (P) is satisfied:

(P) There is a choice mapping $t : X \longrightarrow A$ and a partition $\{B_a\}_{a \in A}$ of the index set B such that: $t(\bigcup_{b \in B_a} R_b) = \bigcup_{b \in B_a} t(R_b) \subset$ $\subset \bigcup_{b \in B_a} I(R_b) = I(\bigcup_{b \in B_a} R_b)$.

Proof of Theorem: For $a \in \omega_\alpha$, put $\tilde{a} = \{f \in S^+(\omega_\alpha) \mid a \in \operatorname{coz} f\}$. Define $\mathcal{P} = \{\tilde{a}\}_{a \in \omega_\alpha}$. Clearly, $\mathcal{P}$ is a uniform cover of $S^+(\omega_\alpha)$. Suppose that there is a uniform cover $\mathcal{Q} = \{Q_\iota\}_{\iota \in \omega_\alpha}$ such that $\mathcal{Q} \overset{+}{<} \mathcal{P}$. By the proposition, there is a $k \geq 2$ such that $\mathcal{P}_k < \mathcal{Q}$. By the Folklore Lemma, there is a choice mapping $t : S^+(\omega_\alpha) \longrightarrow \omega_\alpha$ such that $st(f, \mathcal{P}_k) \subset \widetilde{t(f)}$ for each $f \in S^+(\omega_\alpha)$ and a partition $\{B_a\}_{a \in \omega_\alpha}$ such that the following is satisfied for each $a \in \omega_\alpha$:

(*) $\bigcup_{V \in B_a} t(\mathcal{U}(V)) \subset \bigcap_{V \in B_a} I(\mathcal{U}(V))$ (I was defined in the Folklore Lemma).

Clearly, for each $f \in S^+(\omega_\alpha)$, there is $g \in st(f, \mathcal{P}_k)$ such that $\operatorname{coz} g = f^{-1}(]\!]2^{-k}, 1]\!]) = C^{(f,k)}_{2^k-1}$; hence: (since $st(f, \mathcal{P}_k) \subset \widetilde{t(f)}$):

(**) $t(f) \in C^{(f,k)}_{2^k-1}$.

Define $r : \mathcal{P}(\mathcal{K}^{2^k}(\omega_\alpha)) \longrightarrow \mathcal{P}(\omega_\alpha)$ by $r(\mathcal{D}) = t(\{f \in S^+(\omega_\alpha) \mid C^{(f,k)} \in \mathcal{D}\})$. We obtain from (**) and $C^{(f,k)}_{2^k-1} \subset V_{2^k}$ for each f: (***) $r(\mathcal{W}(V)) \subset V_{2^k}$ for each $V \in \mathcal{K}^{2^k+1}(\omega_\alpha)$. Let $\mathcal{A}$ be a collection of sets whose existence is given by $ZB(\omega_\alpha)$. For $L \in \mathcal{A}$, take a cornet $V^L \in \mathcal{K}^{2^k+1}(\omega_\alpha)$ such that $V^L_{2^k+1} = \omega_\alpha$, $V^L_{2^k} = L$ and $|V_1| = \omega_\alpha$.

It is clear from (**) that $A(2^k, 2^k-1, F, V^L)$ does not hold for any

$F \in [\omega_\alpha]^{\leq \xi}$, so we can use the Basic Lemma: there is $\tilde{V}^L \in \mathcal{K}^{2^k+1}(\omega_\alpha)$ such that $|\tilde{V}^L_{2^k+1}| = \omega_\alpha$, $\tilde{V}^L_{2^k} = L$ and $|r(\mathcal{W}(V^L))| \geq \xi$ (we have used correspondence given by φ_k).

Since there is $f \in \mathcal{W}(\tilde{V}^L)$ such that $\operatorname{coz} f = L$ we also have $I(\mathcal{W}(\tilde{V}^L)) = L$.

Now consider the restriction of $\{B_a\}_{a \in A}$ to $\{\tilde{V}^L | L \in \mathcal{A}\}$. Since $|\mathcal{A}| > \omega_\alpha$, there is a_o such that $|\{L \in \mathcal{A} | \tilde{V}^L \in B_{a_o}\}| \geq |\mathcal{A}| \geq K$.

Denote $\mathcal{A}' = \{L \in \mathcal{A} | \tilde{V}^L \in B_{a_o}\}$. By (*) we have:

$$\bigcup_{L \in \mathcal{A}'} t(\mathcal{W}(\tilde{V}^L)) = \bigcup_{L \in \mathcal{A}'} r(\mathcal{W}(\tilde{V}^L)) \subset \bigcap_{L \in \mathcal{A}'} I(\mathcal{W}(\tilde{V}^L)) = \bigcap_{L \in \mathcal{A}'} L = \bigcap \mathcal{A}' .$$

According to $ZB(\omega_\alpha)$, $|\bigcap \mathcal{A}'| < \xi$, although $|r(\mathcal{W}(\tilde{V}^L))| \geq \xi$ for each $L \in \mathcal{A}$, which is a contradiction.

<u>Comment</u>: We have mentioned that [GCH] implies the negation of $ZB(\omega_\alpha)$. One could doubt whether $ZB(\omega_\alpha)$ is consistent with ZFC. Fortunately, [B] removes these unpleasant questions.

<u>Notation</u>: $AB(K, \lambda, \mu, \nu)$ denotes the assertion: there is $F \subset [K]^\mu$ such that $|F| = \lambda$ and $|X \cap Y| < \nu$ if $X, Y \in F$ and $X \neq Y$.

<u>Theorem Baumgartner</u> [B]: It is consistent with ZFC to suppose that $AB(K, \lambda, K, \nu)$ holds, where $\nu \leq K \leq \lambda$ and ν is regular.

<u>Remark</u>: Clearly, $AB(\omega_\alpha, \omega_\alpha^+, \omega_\alpha, \xi)$, where ξ is a regular cardinal less than $\operatorname{cf} \omega_\alpha$, implies $ZB(\omega_\alpha)$.

So we see that the assertion: "For each uniform space $(X, \mathcal{V})$ and each $\alpha \geq 2$, $p^\alpha(X, \mathcal{V}) = (X, b^\alpha \mathcal{V})$" is consistent with and independent of ZFC.

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