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ENTROPY NUMBERS OF OPERATORS IN BANACH SPACES

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In the following for every operator T between Banach spaces we define a sequence of so-called outer entropy numbers $e_n(T)$ with $n = 1, 2, \ldots$. Roughly speaking the asymptotic behaviour of $e_n(T)$ characterizes the "compactness" of T . In particular, T is compact if and only if $\lim\limits_n e_n(T) = 0$.

The main purpose of this paper is to investigate the ideal $\mathcal{E}_p$ of all operators T such that

$$\sum_1^\infty e_n(T)^p < \infty .$$

For practical reason it is useful to introduce also inner entropy numbers $f_n(T)$ which, however, generate the same ideals.

The concept of entropy numbers is related to that of ε-entropy first studied by L. S. Pontrjagin and L. G. Schnirelman [13] in 1932. Further contributions are mainly due to Soviet mathematicians [1],[2]. For more information the reader is referred to the monograph of G. G. Lorentz [5], see also [4].

The significance of entropy numbers for the theory of operator ideals was discovered by the second named author. A full account will be given in [12].

In the following E , F and G are real Banach spaces. The closed unit ball of E is denoted by U_E . Furthermore, $\mathcal{L}(E, F)$ denotes the Banach space of all (bounded and linear) operators from E into F . The symbols l_p^n and l_p stand for the classical Banach spaces of vectors and sequences, respectively.

All logarithms are to the base 2 .

1. Elementary properties of entropy numbers

For every operator $T \in \mathcal{L}(E, F)$ the n-th <u>outer entropy number</u> $e_n(T)$ is defined to be the infimum of all $\sigma \geq 0$ such that there are $y_1, \ldots, y_q \in F$ with $q \leq 2^{n-1}$ and

$$T(U_E) \subseteq \bigcup_1^q \{y_i + \sigma U_F\} .$$

For every operator $T \in \mathcal{L}(E, F)$ the n-th <u>inner entropy number</u> $f_n(T)$ is defined to be the supremum of all $\rho \geq 0$ such that there are $x_1, \ldots, x_p \in U_E$ with $p > 2^{n-1}$ and

$$\|Tx_i - Tx_k\| > 2\varrho \qquad \text{for } i \neq k .$$

First we state an elementary property of entropy numbers.

Proposition 1.

If $T \in \mathcal{L}(E, F)$, then

$$\|T\| = e_1(T) \geqq e_2(T) \geqq \ldots \geqq 0 \quad \text{and} \quad \|T\| = f_1(T) \geqq f_2(T) \geqq \ldots \geqq 0 .$$

Next we check the so-called additivity of entropy numbers.

Proposition 2.

If $T_1, T_2 \in \mathcal{L}(E, F)$, then

$$e_{n_1+n_2-1}(T_1 + T_2) \leqq e_{n_1}(T_1) + e_{n_2}(T_2)$$

and

$$f_{n_1+n_2-1}(T_1 + T_2) \leqq f_{n_1}(T_1) + f_{n_2}(T_2) .$$

Proof.

Let $T_1, T_2 \in \mathcal{L}(E, F)$. If $\sigma_k > e_{n_k}(T_k)$, then there are $y_1^{(k)}, \ldots, y_{q_k}^{(k)} \in F$ such that

$$T_k(U_E) \subseteq \bigcup_{i=1}^{q_k} \left\{ y_i^{(k)} + \sigma_k U_F \right\} \quad \text{and} \quad q_k \leqq 2^{n_k-1} \quad \text{for } k = 1, 2.$$

Hence, given $x \in U_E$, we can find i_k and $y_k \in U_F$ with

$$T_k x = y_{i_k}^{(k)} + \sigma_k y_k \quad \text{for } k = 1, 2 .$$

It follows from

$$(T_1+T_2)x \in y_{i_1}^{(1)} + y_{i_2}^{(2)} + (\sigma_1 + \sigma_2)U_F$$

that

$$(T_1+T_2)(U_E) \subseteq \bigcup_{i_1=1}^{q_1} \bigcup_{i_2=1}^{q_2} \left\{ y_{i_1}^{(1)} + y_{i_2}^{(2)} + (\sigma_1 + \sigma_2) U_F \right\} .$$

Since $q_1 q_2 \leqq 2^{(n_1+n_2-1)-1}$, we get $e_{n_1+n_2-1}(T_1+T_2) \leqq \sigma_1 + \sigma_2$.

This shows the desired inequality for outer entropy numbers. The remaining part of the proof is left to the reader.

The multiplicativity of entropy numbers can be proved with the same method.

<u>Proposition 3.</u>

If $T \in \mathscr{L}(E, F)$ and $S \in \mathscr{L}(F, G)$, then

$$e_{m+n-1}(ST) \leqq e_m(S)\, e_n(T)$$

and

$$f_{m+n-1}(ST) \leqq f_m(S)\, f_n(T).$$

Finally, the relationship between outer and inner entropy numbers is investigated.

<u>Proposition 4.</u>

If $T \in \mathscr{L}(E, F)$, then

$$f_n(T) \leqq e_n(T) \leqq 2\, f_n(T) .$$

<u>Proof.</u>

Suppose that $\sigma > e_n(T)$ and $\varrho < f_n(T)$. Then we can find $x_1, \ldots, x_p \in U_E$ and $y_1, \ldots, y_q \in F$ with $\|Tx_i - Tx_j\| > 2\varrho$ for $i \neq j$ and $T(U_E) \subset \bigcup_{k=1}^{q} \{y_k + \sigma U_F\}$, where $p > 2^{n-1} \geqq q$. So there must exist different elements Tx_i and Tx_j which belong to the same set $y_k + \sigma U_F$. Consequently $2\varrho < \|Tx_i - Tx_j\| \leqq 2\sigma$. This proves that $f_n(T) \leqq e_n(T)$. Given $\varrho > f_n(T)$, we choose a maximal family of elements $x_1, \ldots, x_p \in U_E$ such that $\|Tx_i - Tx_k\| > 2\varrho$ for $i \neq K$. Clearly $p \leqq 2^{n-1}$. Moreover, for $x \in U_E$ we can find some i with $\|Tx - Tx_i\| \leqq 2\varrho$. This means that

$$T(U_E) \subseteq \bigcup_{1}^{p} \{Tx_i + 2\varrho\, U_F\} .$$

So $e_n(T) \leqq 2\varrho$ and therefore $e_n(T) \leqq 2\, f_n(T)$.

2. <u>Quasi-normed operator ideals related to entropy numbers</u>

In the following let $\mathscr{L}$ denote the class of all operators between Banach spaces while $\mathscr{K}$ denotes the closed ideal of compact operators. Then we have

$$\mathscr{K} = \{T \in \mathscr{L} : (e_n(T)) \in c_0\} .$$

Therefore it seems very natural to introduce the following class of operators. Given $0 < p < \infty$, we define

$$\mathscr{E}_p := \{T \in \mathscr{L} : (e_n(T)) \in l_p\} .$$

Moreover, for $T \in \mathscr{E}_p$ we put

$$E_p(T) := (\sum_1^\infty e_n(T)^p)^{1/p} .$$

We now show that $\mathscr{E}_p$ is a so-called <u>operator ideal</u> for which every component $\mathscr{E}_p(E, F)$ becomes a complete metric linear space with respect to the quasi-norm E_p.

<u>Theorem 1.</u>

If $T_1, T_2 \in \mathscr{E}_p(E, F)$, then $T_1 + T_2 \in \mathscr{E}_p(E, F)$ and

$$E_p(T_1 + T_2) \leqq c\,[E_p(T_1) + E_p(T_2)] ,$$

where

$$c := 2^{1/p} \max (2^{1/p-1}, 1) .$$

<u>Proof.</u>

By Proposition 1 and 2 we get

$$\begin{aligned} E_p(T_1 + T_2) &= \Big\{\sum_1^\infty e_n(T_1 + T_2)^p\Big\}^{1/p} \\ &\leqq \Big\{2 \sum_1^\infty e_{2n-1}(T_1 + T_2)^p\Big\}^{1/p} \\ &\leqq 2^{1/p} \Big\{\sum_1^\infty [e_n(T_1) + e_n(T_2)]^p\Big\}^{1/p} \\ &\leqq c\,[E_p(T_1) + E_p(T_2)] . \end{aligned}$$

<u>Remark.</u>

It follows from Proposition 4 that

$$\mathscr{E}_p = \{T \in \mathscr{L} : (f_n(T)) \in l_p\} .$$

Moreover, by setting

$$F_p(T) := (\sum_1^\infty f_n(T)^p)^{1/p}$$

we define a quasi-norm F_p equivalent to E_p.

Without proof we state

Theorem 2.

If $X \in \mathcal{L}(E_o, E)$, $T \in \mathscr{E}_p(E, F)$ and $Y \in \mathcal{L}(F, F_o)$, then $YTX \in \mathscr{E}_p(E_o, F_o)$ and $E_p(YTX) \leqq \|Y\| \; E_p(T) \; \|X\|$.

The following statement is also evident.

Proposition 5.

If $0 < p_1 < p_2 < \infty$, then $\mathscr{E}_{p_1} \subset \mathscr{E}_{p_2}$ and the embedding map is continuous.

Theorem 3.

If $0 < p, q < \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then $T \in \mathscr{E}_q(E, F)$ and $S \in \mathscr{E}_p(F, G)$ imply that $ST \in \mathscr{E}_r(E, G)$ and $E_r(ST) \leqq$ $\leqq 2^{1/r} E_p(S) \, E_q(T)$.

Proof.

By Proposition 1 and 3 we get

$$\begin{aligned} E_r(ST) &= \left\{ \sum_1^\infty e_n(ST)^r \right\}^{1/r} \\ &\leqq \left\{ 2 \sum_1^\infty e_{2n-1}(ST)^r \right\}^{1/r} \\ &\leqq 2^{1/r} \left\{ \sum_1^\infty \left[e_n(S) \, e_n(T) \right]^r \right\}^{1/r} \\ &\leqq 2^{1/r} E_p(S) \, E_q(T) \, . \end{aligned}$$

This proves the assertion.

3. Quasi-normed operator ideals related to approximation numbers

For every operator $T \in \mathcal{L}(E, F)$ the n-th approximation number is defined by

$$a_n(T) := \inf \left\{ \|T - L\| : L \in \mathcal{L}(E, F) \text{ and rank } (L) < n \right\} .$$

As shown in [9] or [10] the class

$$\mathcal{T}_p := \left\{ T \in \mathcal{L} : \sum_1^\infty a_n(T)^p < \infty \right\}, \; 0 < p < \infty ,$$

is an operator ideal for which every component $\mathcal{T}_p(E, F)$ becomes a complete metric linear space with respect to the quasi-norm

$$S_p(T) := (\sum_1^\infty a_n(T)^p)^{1/p} .$$

Only a little is known about the relationship between $\mathscr{E}_p$ and $\mathscr{T}_p$.

Conjecture 1.

If $0 < p < \infty$, then $\mathscr{T}_p \subseteq \mathscr{E}_p$.

Conjecture 2.

If $0 < p < 2$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$, then $\mathscr{E}_p \subseteq \mathscr{T}_q$.

The inclusions stated above are the best possible which can be expected. Some weaker results are proved in [12].

4. Entropy numbers of operators in Hilbert spaces

It seems to be very complicated to compute or estimate the entropy numbers of a given operator. However, we know some results concerning the related quasi-norms.

Theorem 4.

Let $S \in \mathscr{L}(l_2, l_2)$ such that $S(\xi_n) = (\sigma_n \xi_n)$ and $(\sigma_n) \in$ $\in c_o$. If $\sigma_1 \geqq \sigma_2 \geqq \dots \geqq 0$, then

$$\sigma_n \leqq 2e_n(S) .$$

Proof.

If $\sigma_n = 0$, then the assertion is trivial. So we assume that $\sigma_1 \geqq \sigma_2 \geqq \dots \geqq \sigma_n > 0$. Put

$$J_n(\xi_1, \dots, \xi_n) := (\xi_1, \dots, \xi_n, 0, \dots)$$

and

$$Q_n(\xi_1, \dots, \xi_n, \xi_{n+1}, \dots) := (\xi_1, \dots, \xi_n) .$$

Then $S_n = Q_n S J_n$ is invertible. If I_n denotes the identity map of l_2^n , it follows from $e_n(I_n) \geqq 1/2$ and Proposition 2 that

$$1/2 \leqq e_n(I_n) \leqq e_n(S_n) \|S_n^{-1}\| \leqq \|Q_n\| \quad e_n(S) \|J_n\| \sigma_n^{-1}$$

$$\leqq e_n(S)\ \sigma_n^{-1} .$$

This completes the proof.

Theorem 5.

Let $S \in \mathscr{L}(l_2, l_2)$ such that $S(\xi_n) = (\sigma_n \xi_n)$ and $(\sigma_n) \in$ $\in c_o$. Then

$$(\sum_1^\infty e_n(S)^p)^{1/p} \leqq c_p(\sum_1^\infty |\sigma_n|^p)^{1/p} \quad \text{for} \quad 0 < p < \infty ,$$

where c_p is some positive constant.

Proof.

Without loss of generality we may suppose that $\sigma_1 \geqq \sigma_2 \geqq \ldots$ $\ldots \geqq 0$. Let

$$E(\varepsilon) := \max\{n : e_n(S) > \varepsilon\} \qquad \text{for} \quad 0 < \varepsilon < \sigma_1 .$$

We now show that

$$(*) \qquad E(2\varepsilon) \leqq 1 + \sum_{\sigma_k > \varepsilon} \log (8\,\sigma_k/\varepsilon) .$$

Put $m := \max (k : \sigma_k > \varepsilon)$ and $S_m := Q_m S J_m$. Let U_2^m and U_∞^m denote the closed unit ball of l_2^m and l_∞^m, respectively. If $y \in S_m(U_2^m)$, then there exists $g = (\gamma_1, \ldots, \gamma_m)$ such that

$$y \in \varepsilon m^{-1/2} \{2g + U_\infty^m\} \subseteq 2\varepsilon\ m^{-1/2} g + \varepsilon U_2^m ,$$

where $\gamma_1, \ldots, \gamma_m$ are integers. Since $\sigma_1 \geqq \ldots \geqq \sigma_m > \varepsilon$, we have

$$\varepsilon m^{-1/2} \{2g + U_\infty^m\} \subseteq S_m(U_2^m) + 2\varepsilon m^{-1/2} U_\infty^m \subseteq 3S_m(U_2^m) .$$

Let $g_1, \ldots, g_q$ be the collection of all $g_i = (\gamma_{i1}, \ldots, \gamma_{im})$ with

$$\varepsilon m^{-1/2} \{2g_i + U_\infty^m\} \subseteq 3\ S_m(U_2^m) .$$

Clearly

$$S_m(U_2^m) \subseteq \bigcup_1^q \{2\varepsilon m^{-1/2} g_i + \varepsilon U_2^m\}$$

and therefore

$$S(U_2) \subseteq J_m\ S_m(U_2^m) + \sigma_{m+1} U_2 \subseteq \bigcup_1^q \{2\varepsilon m^{-1/2}\ J_m g_i + 2\varepsilon U_2\},$$

where U_2 denotes the closed unit ball of l_2 . On the other hand,

$$q\,[\varepsilon m^{-1/2}]^m\ \lambda(U_\infty^m) = \sum_1^q \lambda[\varepsilon m^{-1/2}\{2g_i + U_\infty^m\}] \subseteq 3^m \prod_1^m \sigma_k\ \lambda(U_2^m),$$

where λ is the Lebesgue measure. Using Stirling's formula we get

$$e^t\Gamma(t+1) \geqq \sqrt{2\pi}\; t^{t+\frac{1}{2}} \quad \text{for} \quad 0 < t < \infty .$$

Hence

$$\lambda(U_2^m) = \frac{\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2}+1)} \leqq \frac{(2\pi e)^{\frac{m}{2}}}{m^{m/2}} \leqq \frac{5^m}{m^{m/2}} .$$

This implies that

$$q\,\varepsilon^m \leqq 8^m \prod_1^m \sigma_k .$$

Choose n such that

$$n-1 \leqq 1 + \sum_1^m \log(8\,\sigma_k/\varepsilon) \leqq n .$$

Then $q \leqq 2^{n-1}$ and therefore $e_n(S) \leqq 2\varepsilon$. So $E(2\varepsilon) \leqq n-1$. This proves $(\ast)$.

Finally, we have

$$\begin{aligned}
2^{-p}\sum_1^\infty e_n(S)^p &= 2^{-p}\sum_1^\infty n\left[e_n(S)^p - e_{n+1}(S)^p\right] \\
&= 2^{-p}\int_0^{\sigma_1} E(\varepsilon)\, d\varepsilon^p \\
&\leqq \sigma_1^p + \int_0^{\sigma_1} \sum_{\sigma_k>\varepsilon} \log(8\,\sigma_k/\varepsilon)\, d\varepsilon^p \\
&= \sigma_1^p + \sum_{i=1}^\infty \int_{\sigma_{i+1}}^{\sigma_i} \sum_{\sigma_k>\varepsilon} \log(8\,\sigma_k/\varepsilon)\, d\varepsilon^p \\
&= \sigma_1^p + \sum_{i=1}^\infty \sum_{k=1}^{i} \int_{\sigma_{i+1}}^{\sigma_i} \log(8\,\sigma_k/\varepsilon)\, d\varepsilon^p \\
&= \sigma_1^p + \sum_{k=1}^\infty \sum_{i=k}^{\infty} \int_{\sigma_{i+1}}^{\sigma_i} \log(8\,\sigma_k/\varepsilon)\, d\varepsilon^p \\
&= \sigma_1^p + \sum_{k=1}^\infty \int_0^{\sigma_k} \log(8\,\sigma_k/\varepsilon)\, d\varepsilon^p \\
&= \sigma_1^p + \frac{8^p}{p}\int_0^{8^{-p}} \log(1/t)\,dt \sum_1^\infty \sigma_k^p .
\end{aligned}$$

This completes the proof.

The above theorems show that for any Hilbert space H the operator ideal $\mathscr{C}_p(H, H)$ coincides with the operator ideal $\mathscr{T}_p(H,H)$. In particular, $\mathscr{C}_2(H, H)$ is the ideal of so-called Hilbert-Schmidt Operators.

5. Entropy quasi-norms of the identity map I_n from l_u^n to l_v^n

Lemma 1.

If $m = 1, \ldots, n$, then

$$e_m(I_n : l_\infty^n \to l_1^n) \geqq \frac{1}{2e} n .$$

Proof.

Let U_∞^n and U_1^n denote the closed unit ball of l_∞^n and l_1^n, respectively. Suppose that

$$U_\infty^n \subseteq \bigcup_1^q \{y_i + \sigma U_1^n\} \text{ and } q \leqq 2^{n-1} .$$

Then

$$\lambda(U_\infty^n) \leqq \sum_1^q \lambda(y_i + \sigma U_1^n) = q \, \sigma^n \, \lambda(U_1^n) ,$$

where λ is the Lebesgue measure on R^n . Now $\lambda(U_\infty^n) = 2^n$ and $\lambda(U_1^n) = 2^n/n!$ imply that $\sigma^n \geqq n!/2^{n-1}$. Using $e^n n! > n^n$ we get $\sigma > n/2e$. Therefore

$$e_n(I_n : l_\infty^n \to l_1^n) \geqq n/2e .$$

In order to prove the following lemma we use a decomposition-trick taken from M. Š. Birman and M. Z. Solomjak [1].

Lemma 2.

If $m = 1, \ldots, n$, then

$$e_m(I_n : l_1^n \to l_\infty^n) \leqq c \, \frac{\log(n+1)}{m} ,$$

where c is a positive constant.

Proof.

Let U_1^n and U_∞^n be as before. If $m \geqq 4$, then

$$\sigma := 4 \, \frac{\log(n+1)}{m} \geqq 2 \, \frac{\log(n+1)}{m-2} > \frac{1}{n} .$$

Put

$$K(x) := \{ k : |\xi_k| > \sigma \} \quad \text{for} \quad x = (\xi_k) \in U_1^n .$$

We have

$$\text{card}\,(K(x)) < \sum_{K(x)} \frac{|\xi_k|}{\sigma} \leqq 1/\sigma < n .$$

Let $\mathbb{K}$ denote the collection of all sets $K \subseteq \{1, \ldots, n\}$ with card $(K) < 1/\sigma$ and put

$$U_k := \{x \in U_\infty^n \; : \; \xi_k = 0 \text{ if } k \notin K\}.$$

Then

$$x \in U_{K(x)} + \sigma U_\infty^n \quad \text{for all} \quad x \in U_1^n .$$

Hence

$$U_1^n \subseteq \bigcup_{\mathbb{K}} \{U_K + \sigma U_\infty^n\}.$$

Clearly, we can find $y_i^{(K)} \in l_\infty^n$ such that

$$U_K \subseteq \bigcup_1^{q_K} \{y_i^{(K)} + \sigma U_\infty^n\} \quad \text{and} \quad q_K \leqq (1/\sigma + 1)^{\operatorname{card}(K)} .$$

Consequently, there are $y_i \in l_\infty^n$ with

$$U_1^n \subseteq \bigcup_1^q \{y_i + 2 \sigma U_\infty^n\}$$

and

$$q \leqq \sum_{\mathbb{K}} (1/\sigma + 1)^{\operatorname{card}(K)} \leqq \sum_1^{1/\sigma} \binom{n}{h} (1/\sigma + 1)^h \leqq 2(n+1)^{2/\sigma} \leqq 2^{m-1}.$$

So we get

$$e_m(I_n : l_1^n \rightarrow l_\infty^n) \leqq 2 \sigma \leqq 8 \frac{\log(n+1)}{m} .$$

Obviously this estimate is also true for $m = 1, 2, 3$.

<u>Proposition 6.</u>

If $0 < p < \infty$, then

$$E_p(I_n : l_\infty^n \rightarrow l_1^n) \geqq a_p \; n^{1/p+1} \quad \text{for} \quad n = 1,2,\ldots,$$

where a_p is some positive constant.

<u>Proof.</u>

By Lemma 1 we have

$$e_m(I_n : l_\infty^n \rightarrow l_1^n) \geqq \frac{1}{2e} n \quad \text{for} \quad m = 1, \ldots, n .$$

Therefore

$$E_p(I_n : l_\infty^n \rightarrow l_1^n) \geqq \frac{1}{2e} n^{1/p+1} .$$

<u>Proposition 7.</u>

If $0 < p < 1$, then

$$E_p(I_n : l_1^n \to l_\infty^n) \leq b_p\, n^{1/p-1} \log(n+1) \quad \text{for} \quad n = 1, 2, \ldots,$$

where b_p is some positive constant.

Proof.

Using Proposition 3 we have

$$E_p(I_n : l_1^n \to l_\infty^n) \leq \Big(\sum_{k=0}^{\infty} \sum_{m=1}^{n} e_{kn+m}(I_n : l_1^n \to l_\infty^n)^p\Big)^{1/p}$$

$$\leq \Big(\sum_{m=1}^{n} e_m(I_n : l_1^n \to l_\infty^n)^p\Big)^{1/p} \Big(\sum_{k=0}^{\infty} e_{n+1}(I_n : l_\infty^n \to l_\infty^n)^{kp}\Big)^{1/p}.$$

From $e_{n+1}(I_n : l_\infty^n \to l_\infty^n) = 1/2$ we get

$$\Big(\sum_{k=0}^{\infty} e_{n+1}(I_n : l_\infty^n \to l_\infty^n)^{kp}\Big)^{1/p} \leq c_p\ .$$

By Lemma 2 it follows that

$$\Big(\sum_{m=1}^{n} e_m(I_n : l_1^n \to l_\infty^n)^p\Big)^{1/p} \leq d_p\, n^{1/p-1} \log(n+1)\ .$$

Since the constants c_p and d_p do not depend on n, the assertion is proved.

Theorem 6.

If $0 < p < 1$ and $1 \leq u, v \leq \infty$, then

$$a_p n^{1/p+1/v-1/u} \leq E_p(I_n : l_u^n \to l_v^n) \leq b_p n^{1/p+1/v-1/u} \log(n+1) \quad \text{for} \quad n = 1, 2, \ldots,$$

where a_p and b_p are positive constants.

Proof.

By Theorem 2 and Proposition 6 we get

$$a_p n^{1/p+1} \leq E_p(I_n : l_\infty^n \to l_1^n) \leq \|I_n : l_\infty^n \to l_u^n\| E_p(I_n : l_u^n \to l_v^n)\ \|I_n : l_v^n \to l_1^n\|$$

$$\leq n^{1/u} E_p(I_n : l_u^n \to l_v^n)\, n^{1-1/v}$$

and therefore

$$a_p\, n^{1/p+1/v-1/u} \leq E_p(I_n : l_u^n \to l_v^n)\ .$$

Analogously, by Theorem 2 and Proposition 7 we have

$$E_p(I_n : l_u^n \to l_v^n) \leq \|I_n : l_u^n \to l_1^n\|\ E_p(I_n : l_1^n \to l_\infty^n)\ \|I_n : l_\infty^n \to l_v^n\|$$

$$\leq n^{1-1/u} b_p n^{1/p-1} \log(n+1)\, n^{1/v} =$$

$$= b_p\, n^{1/p+1/v-1/u} \log(n+1)\ .$$

The limit order $\lambda(\mathscr{E}_p, u, v)$ is defined to be the infimum of all $\lambda \geqq 0$ such that

$$E_p(I_n : l_u^n \to l_v^n) \leqq c\, n^{\lambda} \qquad \text{for } n = 1, 2, \dots ,$$

where c is some constant. Using this concept we can restate the above result as follows.

Theorem 7.

If $0 < p < 1$ and $1 \leqq u, v \leqq \infty$, then

$$\lambda(E_p, u, v) = 1/p+1/v-1/u\ .$$

The remaining case is treated in the next theorem. For the proof the reader is referred to [12].

Theorem 8.

If $1 \leqq p < \infty$ and $1 \leqq u\ , v \leqq \infty$, then

$$\lambda(E_p, u, v) = \max\,(1/p+1/v-1/u,\ 0).$$

The limit order is very useful for formulating conditions for a given diagonal operator $S(\xi_n) = (\sigma_n \xi_n)$ to belong to $\mathscr{E}_p(l_u, l_v)$. According to a deep theorem of H. König (3) our results can also be carried across to embedding maps of Sobolev spaces and to weakly singular integral operators from L_u into L_v .

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