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CATEGORY, BOOLEAN ALGEBRAS AND MEASURE

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Introduction

It should be said at once that the "category" in the title refers to Baire category. A topological measure space X will have three naturally-arising complete Boolean algebras: the algebras $\mathcal{A}_r(X)$ of regular open sets, $\mathcal{A}_c(X)$ of Borel sets modulo first category sets, and $\mathcal{A}_m(X)$ of measurable sets modulo null sets. While $\mathcal{A}_m(X)$ is obviously the algebra of greatest interest to analysts, $\mathcal{A}_c(X)$ (the "category algebra" of X in the terminology of Oxtoby [10]) is also of considerable interest to them. It turns out, however, that $\mathcal{A}_c(X)$ does not behave very well (under product formation, for instance) unless X is "nice", in which case $\mathcal{A}_c(X)$ is the same as $\mathcal{A}_r(X)$. Thus it pays to prove general theorems about the better-behaved $\mathcal{A}_r(X)$ rather than $\mathcal{A}_c(X)$, even though one may be more interested in the latter.

Accordingly we begin by discussing $\mathcal{A}_r$, in § 1. In § 2 and § 3 we consider $\mathcal{A}_c$ and $\mathcal{A}_m$ respectively. In §§ 4-6 we compare and contrast the behavior of $\mathcal{A}_c$ and $\mathcal{A}_m$ with respect to problems concerning liftings, completions and mappings from representation spaces. Finally in § 7 we apply some of the results of previous sections to construct a "completion" for C(X). The unifying thread connecting these topics is simply that they have arisen in the course of the author's recent work. Much of what follows may well be known, but (apart from the references given below) I have not found most of it in the literature.

I am grateful to A. H. Stone for some helpful discussions.

1. The regular open algebra

For an arbitrary topological space X we write $\mathcal{A}_r(X)$ for the family of regular open subsets of X; this (ordered by set inclusion) is well known to be a complete Boolean algebra, the supremum of a family of regular open sets being the interior of the closure of their union. (The infimum of a <u>finite</u> number of regular open sets is their intersection.) We note that <u>every</u> complete Boolean algebra $\mathcal{A}$ arises as the regular open algebra of some (compact, Hausdorff, extremally disconnected) space X - namely, the Stone representation space of $\mathcal{A}$, which we shall denote by $R(\mathcal{A})$. (See [13, p. 117].)

We say that two spaces X , Y are "regular open equivalent",and write $X \sim_r Y$, to mean that $\mathcal{A}_r(X)$ and $\mathcal{A}_r(Y)$ are isomorphic. Thus, for example, $X \sim_r R(\mathcal{A}_r(X))$ for all X . Oxtoby [10] has given a method for constructing all spaces Y for which $X \sim_r Y$; but it is (inevitably) not easily applied in particular cases - for, from the above remark, an effective method here would imply the classification of all complete Boolean algebras. However, Oxtoby obtains a striking consequence (though it is easily proved directly):

(1) If D is a dense subset of X , then $D \sim_r X$.

As this shows, r-equivalent spaces can be very different topologically. Thus, for example, if X is a Tychonoff space, then $X \sim_r \beta X$. Again, let K denote $\{0\} \cup \{n^{-1}: n \in N\}$, where N is the set of positive integers; then $K \sim_r \beta N$ (because both have dense subsets homeomorphic to N), despite the disparity in their cardinals, and despite the fact that both are compact.

However, r-equivalence does preserve some topological properties. The following instances were obtained jointly by A.H. Stone and myself. First, the "density character" $\delta(X)$ is defined as usual as the smallest cardinal of a dense subset of X . Define the "dense density character" $\delta\delta(X)$ to be the smallest cardinal d such that every dense subset of X has a dense subset of cardinal $\leq d$.(Clearly $\delta(X) \leq \delta\delta(X)$; there need not be equality.) Then:

(2) If $X \sim_r Y$ then $\delta(X) = \delta(Y)$.

(3) If $X \sim_r Y$ and X, Y are compact Hausdorff, then $\delta\delta(X) = \delta\delta(Y)$.

(4) If X and Y are non-empty separable metric spaces (or, more generally, are regular T_1 first countable spaces) without isolated points, then $X \sim_r Y$.

For X and Y , in (4), will have dense subspaces homeomorphic to the space Q of rationals, as follows from [12]. Note that (4) applies, for example, to the Sorgenfrey line and plane.

To formulate a more inclusive result that allows for isolated points, write $\mathcal{I}(X)$ = the set of all isolated points of X , $\mathcal{D}(X)$ = = $X - Cl(\mathcal{I}(X))$. Note that the isolated points of a regular T_1 space X are precisely the atoms of $\mathcal{A}_r(X)$, and that $\mathcal{D}(X)$ is precisely the complement, in $\mathcal{A}_r(X)$, of their supremum. Hence a regular open equivalence between X and Y induces a one-one correspondence between $\mathcal{J}(X)$ and $\mathcal{J}(Y)$, and also a regular open equivalence between $\mathcal{D}(X)$ and $\mathcal{D}(Y)$. Conversely, we have (taking the metrizable case for simplicity of statement, and writing $|E|$ for the cardinal of E):

(5) If X and Y are separable metric spaces, and if $|\mathcal{J}(X)| = |\mathcal{J}(Y)|$ and $\mathcal{D}(X)$, $\mathcal{D}(Y)$ are either both empty or both non-empty, then $X \sim_r Y$.

For, as in (4), $\mathcal{D}(X)$ and $\mathcal{D}(Y)$ will have homeomorphic dense subspaces, and these, together with $\mathcal{J}(X)$ and $\mathcal{J}(Y)$, provide regular open equivalent dense subspaces of X and Y, to which we apply (1).

A fairly straightforward argument will also prove:

(6) If $X_\lambda \sim_r Y_\lambda$ (for all $\lambda \in \Lambda$) then $\prod_\lambda X_\lambda \sim_r \prod_\lambda Y_\lambda$.

From this and the foregoing we see that, for example, if k is any uncountable cardinal, the spaces 2^k, N^k, R^k, I^k, $(\beta N)^k$ are all regular open equivalent.

Another way of looking at these results comes from the fact that in these cases (and some others) it is possible to give fairly simple characterizations of the Boolean algebras $\mathcal{A}_r(X)$. For instance, if X is as in (4) - we may as well say $X = I$, the unit interval - then $\mathcal{A}_r(X)$ is characterized, to within isomorphism, as being a complete non-atomic Boolean algebra with a countable σ-basis (see [1, p. 177]). From this a (more complicated) characterization of $\mathcal{A}_r(I^k)$ can be derived. Of course, if X has a dense discrete subset D, $\mathcal{A}_r(X)$ is isomorphic to the algebra $\mathcal{P}(D)$ of all subsets of D, for which characterizations are also known [14]. And in (5), $\mathcal{A}_r(X)$ is characterized (if $\mathcal{D}(X) \neq \emptyset$) as the direct sum of $\mathcal{A}_r(I)$ and $\mathcal{P}(\mathcal{J}(X))$.

Finally we mention the easily verified fact:

(7) If $f : X \to Y$ is a continuous open surjection, then f^{-1} gives an isomorphism of $\mathcal{A}_r(Y)$ onto a complete subalgebra of $\mathcal{A}_r(X)$.

2. The category algebra

Again let X be a topological space, $\mathcal{B}$ its family of Borel sets, $\mathcal{C}$ its family of sets of first category (in X). Let $\mathcal{B} + \mathcal{C}$ denote the family of sets differing from Borel sets by sets of first category. The "category algebra" $\mathcal{A}_c(X)$ is defined to be the quotient algebra $(\mathscr{B} + \mathscr{C})/\mathscr{C}$. As is well known, there is a natural homomorphism $f : \mathscr{A}_r(X) \to \mathscr{A}_c(X)$, which is an isomorphism if, and only if, X is a Baire space (that is, no non-empty open subset of X is of first category in itself - or, equivalently, in X). We define $X \sim_c Y$ to mean that $\mathscr{A}_c(X)$ and $\mathscr{A}_c(Y)$ are isomorphic. Thus, for locally compact Hausdorff spaces, and for complete metric spaces, $\mathscr{A}_r(X) = \mathscr{A}_c(X)$, and $\sim_c$ coincides with $\sim_r$.

In general, there is no implication between $\sim_c$ and $\sim_r$. For example, $Q \sim_r R$, by 1.(4), but $\mathscr{A}_c(Q) = \{0\} \neq \mathscr{A}_c(R)$. Again, $\{0\} \sim_c Q$, but $\mathscr{A}_r(\{0\}) \neq \mathscr{A}_r(Q)$. Nevertheless there is a sense in which the category algebra is reducible to the regular open algebra (and $\sim_c$ to $\sim_r$). For, given a space X, the union U^* of all its open sets of first category is, by a theorem of Banach [6, p. 82], also an open set of first category. Put $X^* = X - \overline{U^*}$; then X^* is a Baire space, and $\mathscr{A}_c(X)$ is isomorphic to $\mathscr{A}_c(X^*) = \mathscr{A}_r(X^*)$.

This shows that the assumption we shall usually make, when studying the category algebra, that the spaces involved are Baire, is not an enormous one. It enables us to transfer the results of the previous section to $\mathscr{A}_c$ and $\sim_c$; for instance, 1.(1) says that if D is a dense subset of X, and both D and X are Baire spaces (it suffices that D is Baire), then $D \sim_c X$. Of course, 1.(3) applies to $\sim_c$ as it stands. Note that the analogue of 1.(6) is complicated by the need to require that the product spaces too are Baire sets, which in general they need not be ([11], [16]).

Not every significant property of $\sim_c$ arises as a special case of one of $\sim_r$. A topological space X is said to be "residually Lindelöf" if every open cover $\mathcal{U}$ of X has a countable subsystem $U_1, U_2, \ldots$, such that $X - \bigcup_{n=1}^{\infty} U_n$ is of first category (see [5]). Say that X is "hereditarily residually Lindelöf" if every open subset of X is residually Lindelöf. Then we have:

(1) Suppose X and Y are Baire spaces and $X \sim_c Y$. Then if X is hereditarily residually Lindelöf, so is Y.

(More generally, an analogous definition can be given for "hereditarily residually $(\alpha - \beta)$ compact", and the analogous result will hold.) Thus, for instance, every open subset of $R(\mathscr{A}_c(I))$ will be residually Lindelöf. Note that the analogue of (1) for not necessarily Baire spaces such that $X \sim_r Y$ would be false - for instance when $X = R \times D$ and $Y = Q \times D$ with D an uncountable discrete space.

3. The measure algebra

Now assume that the topological space X also has a finite (or σ-finite) regular Borel measure μ ; that is, μ is a non-negative , countably additive measure defined on the family $\mathcal{B}$ of Borel sets of X , with the property that $\mu(B) = \inf\{\mu(G) : G \text{ is open and } G \supset B\}$ for each $B \in \mathcal{B}$. Put $\mathcal{N} = \{E \subset X : \text{there exists } B \in \mathcal{B} \text{ such that } B \supset E$ and $\mu(B) = 0\}$. Then μ extends in the obvious way to the family $\mathcal{B} + \mathcal{N}$ of sets that differ from Borel sets by members of $\mathcal{N}$. We put $\mathcal{A}_m = (\mathcal{B} + \mathcal{N})/\mathcal{N}$. This too is a complete Boolean algebra, and it presents some analogies with $\mathcal{A}_c$. For instance, we can without much loss require that $\mu(G)$ be positive for every non-empty open set G, by replacing X by the complement of the union of all open sets of measure 0 (this union is of measure 0 because, since μ is σ-finite, $\mathcal{A}_m$ satisfies the countable chain condition). This would be the analogue of replacing X by the Baire space X^* in the previous section. Nevertheless there are some sharp differences between $\mathcal{A}_m$ on the one hand, and $\mathcal{A}_c$ and $\mathcal{A}_r$ on the other. Like $\mathcal{A}_r$, $\mathcal{A}_c$ can hardly be expected to have a simple explicit structure theory, for that would amount to a structure theory for all complete Boolean algebras. But $\mathcal{A}_m$ has a reasonably satisfactory structure theory (independent of the topological assumptions), as follows. Write $X \sim_m Y$ to mean that $\mathcal{A}_m(X)$ and $\mathcal{A}_m(Y)$ are isomorphic. Then [7] given X we have $X \sim_m Y$ where Y is the discrete union of countably many measure spaces, each of which is either an atom or (to within a constant scaling factor) a product I^k of copies of the unit interval I , with product Lebesgue measure.

Another difference is that $\mathcal{A}_m$ has the property (a consequence of the regularity of μ and of Urysohn's Lemma):

(1) If X is normal (qua topological space), each measure class contains a Baire set.

(Here, as usual, the Baire sets are the σ-field generated by the zero-sets.) The analogue of (1) for $\mathcal{A}_c$ is false, in general, even for compact Hausdorff spaces, as is shown by the following example (pointed out to me by A. H. Stone). Take X to be the usual space of ordinals $\leq \omega_1$, and split the non-limit ordinals into two complementary cofinal sets, say E and F . Both E and F are open, hence Borel; but neither can differ from a Baire set by a first category set.

Nevertheless, in every product of separable metric spaces (I^k, for instance) it can be shown that every regular open set <u>is</u> a Baire set; thus in this case each category class (of a Borel set modulo first category) does contain a Baire set. It would be interesting to know (a) for what spaces every Borel set differs from a Baire set by a set of first category, (b) for what spaces all regular open sets are Baire (or, more specifically, are co-zero).

We observe that, in $\mathscr{A}_c(X)$, each category class a contains a largest open set G(a) (namely, the union of all open sets in the class) and a smallest closed set F(a). If X is a Baire space, G(a) is the unique regular open set in a , and F(a) is the unique regular closed set in a , and we have $F(a) = Cl(G(a)), G(a) = Int(F(a))$. The analogue for $\mathscr{A}_m(X)$ fails; in general, a measure class a will contain neither an open set nor a closed set. However, <u>if</u> the measure class a contains an open set, it contains a largest one, say $G_1(a)$, and we call a an "open class". Similarly a "closed class" a is one that contains a closed set, and hence a smallest closed set, say $F_1(a)$. The "<u>ambiguous classes</u>" are defined to be those that are both open and closed. (This notion has been considered independently by S. Graf, in unpublished work.) If we assume (without essential loss, as remarked above) that each non-empty open set in X has positive measure, then we have $F_1(a) = Cl(G_1(a))$ and $G_1(a) = Int(F_1(a))$ for all ambiguous classes a , in analogy with the situation in $\mathscr{A}_c(X)$. We shall make use of the ambiguous measure classes in § 6 below.

4. <u>Liftings</u>

Suppose $\mathscr{E}$ is an arbitrary Boolean algebra, and $\mathscr{J}$ is an arbitrary ideal in $\mathscr{E}$. Let $\mathscr{A}$ be the factor algebra $\mathscr{E}/\mathscr{J}$. A "lifting" of $\mathscr{A}$ is a homomorphism h of $\mathscr{A}$ into $\mathscr{E}$ (qua finitely additive Boolean algebras; h need not preserve infinite operations, even if they are available), such that $h(A) \in a$ for all $a \in \mathscr{A}$. Suppose in particular that $\mathscr{E}$ is an algebra of subsets of a space X ; then a "strong lifting" is one with the property that whenever $G \in \mathscr{E}$ is an open set in X with g (say) as its class mod $\mathscr{J}$, then h(g) is an open set containing G .

The following theorem seems to be generally known, though I have not seen it in print in exactly this form. It follows easily from a theorem of Graf [4]; and independent, unpublished proofs have been obtained by J. P. R. Christensen and by myself.

(1) If X is a Baire space, the category algebra $\mathcal{A}_c(X) = (\mathcal{B} + \mathcal{C})/\mathcal{C}$ always has a strong lifting.

The proof of (1) is basically a Zorn's Lemma argument, taking the representative $h(a)$ to be intermediate between $G(a)$ and $F(a)$, in the notation of the previous section. It is (so far as I know) an open question whether one can always take $h(a)$ to be a Borel set.

Analogously, $\mathcal{A}_m = (\mathcal{B} + \mathcal{N})/\mathcal{N}$ always has a lifting [8]; the roles of $G(a)$ and $F(a)$ in the preceding are now taken by the sets of upper and lower density. Again, it is (so far as I know) an open question whether there is always a <u>strong</u> lifting for $\mathcal{A}_m$ (say if X is compact Hausdorff), assuming of course that the measure of every non-empty open set is positive. Perhaps the study of the "ambiguous classes" of § 3 may throw some light on this.

In the same order of ideas, we can ask under what conditions an automorphism h of $\mathcal{A} = \mathcal{E}/\mathcal{J}$, where $\mathcal{E}$ is an algebra of subsets of X, can be "realized" by a suitable point-transformation $f : X \to X$ (so that $h(a)$ is in the class of $f(E)$ for every E in the class a). We are concerned here with the cases $\mathcal{A} = \mathcal{A}_c(X)$ or $\mathcal{A}_m(X)$. Even for these, easy counterexamples show that X will have to be very special; "compact Hausdorff" is not enough. However, Choksi has shown [2] that when X is a compact Hausdorff group, then every automorphism of $\mathcal{A}_m(X)$ can be realized by a (both-ways measurable) bijection of X onto itself. On the other hand, not every automorphism of $\mathcal{A}_c(C)$, where C is the Cantor set, can be realized by a <u>homeomorphism</u>. The following provides an example. Choose a 2-sided limit point $\alpha \in C$, and put

$A = [0,\alpha] \cap C$, $B = [\alpha, 1] \cap C$, $U = [0, 1/3] \cap C$, $V = [2/3, 1] \cap C$.

Then $\mathcal{A}_c(U) = \mathcal{A}_r(U)$ and $\mathcal{A}_c(A) = \mathcal{A}_r(A)$ are isomorphic, and $\mathcal{A}_c(V)$, $\mathcal{A}_c(B)$ are isomorphic; and these isomorphisms combine to give an automorphism of $\mathcal{A}_c(C)$ that takes the class of U to the class of A. If this could be realized by a homeomorphism h, then $h(U)$ and A would be regular closed sets in the same category class, and would therefore coincide; but $h(U)$ is open, and A is not. It can be shown, however, that <u>every automorphism of $\mathcal{A}_c(C)$ can be realized by a homeomorphism of a dense G_δ subset of C onto itself</u>. This answers a question asked me by S. Kakutani, in conversation. I hope to publish the proof elsewhere.

5. Completions

Let $\mathscr{F}$ be an arbitrary Boolean algebra; consider its representation space $R(\mathscr{F})$, and put $\mathscr{F}^* = \mathscr{A}_c(R(\mathscr{F}))$ $(= \mathscr{A}_r(R(\mathscr{F})))$. Then $\mathscr{F}$, qua finitely additive algebra, is a subalgebra of the complete algebra $\mathscr{F}^*$ (that is, the natural embedding of $\mathscr{F}$ in $\mathscr{F}^*$ preserves finite infs and sups, but not in general infinite ones, even when they are available). Roughly speaking, $\mathscr{F}^*$ is the smallest complete algebra containing $\mathscr{F}$ in this sense; this is the content of the following theorem, which follows easily from one in [13, p. 141]:

(1) If Θ is an isomorphism (finitely additive) of $\mathscr{F}$ into a complete Boolean algebra $\mathscr{G}$, then there is a unique extension of Θ to an isomorphism Θ^* of $\mathscr{F}^*$ onto a (finitely additive) subalgebra of $\mathscr{G}$.

(Here $\Theta^*(\mathscr{F}^*)$, though itself necessarily a complete algebra, is guaranteed only to have its finite operations agree with those of $\mathscr{G}$.)

Now suppose μ is a finitely additive (non-negative, finite) measure on $\mathscr{F}$. Then μ extends to a countably additive measure on the family $\mathscr{B}$ of Borel sets of $R(\mathscr{F})$; and the corresponding measure algebra $\mathscr{G}$ is a complete Boolean algebra, to which (1) applies. This (with some elementary considerations) proves:

(2) μ has a unique extension to a finitely additive measure μ^* on $\mathscr{F}^*$; further, μ^* is reduced (that is, vanishes only for the zero element) if, nad only if, μ is.

It follows, for example, that

(3) there exists a finitely additive, finite reduced measure on $\mathscr{A}_c(I^k)$, where k is an arbitrary infinite cardinal.

For $\mathscr{A}_c(I^k) = \mathscr{A}_c(2^k)$ by 1.(6). Let $\mathscr{F}$ denote the finitely additive algebra formed by the open-closed sets in 2^k. The restriction of the usual Lebesgue product measure to $\mathscr{F}$ gives a suitable μ to which (2) applies. Here $R(\mathscr{F}) = 2^k$, and therefore $\mathscr{F}^* = \mathscr{A}_c(2^k)$.

Note that $\mathscr{A}_c(I^k)$ does not carry a countably additive reduced, finite measure [1, p. 186].

It would be good to have a structure theory for finitely additive measures similar to that (described in § 3) for countably additive ones; but this will not be easy. One conjecture might be that such a finitely additive measure algebra - say with a reduced, non-atomic, finite measure μ - might be isomorphic to a direct sum of terms of the form $\mathcal{A}_c(2^k)$, each with a suitable (finitely additive) measure. Unfortunately this is false, because it can be shown that this would imply that the "density measures" on $\mathcal{P}(N)$ (see [9]) would have liftings; and they don't.

6. Spaces as continuous images of representation spaces

Let X be a compact Hausdorff space, and denote by $\tilde{X}$ the representation space $R(\mathcal{A}_c(X))$ ($= R(\mathcal{A}_r(X))$). Gleason has observed [3] that the natural isomorphism between $\mathcal{A}_c(X)$ and $\mathcal{A}_c(\tilde{X})$ can be realized by a continuous surjection $\Theta : \tilde{X} \to X$. In fact, one can define, for each $\tilde{\alpha} \in \tilde{X}$ (so that $\tilde{\alpha}$ is an ultrafilter on $\mathcal{A}_c(X)$)

$$\{\Theta(\tilde{\alpha})\} = \bigcap \{F(a) : a \in \tilde{\alpha}\} ,$$

where (as in § 3) F(a) is the smallest (regular) closed set in the class $a \in \mathcal{A}_c(X)$. Of course, it has to be checked (among other things) that this intersection really is a singleton.

An analogous theorem holds for the measure-algebraic case. Suppose μ is a measure on X , as in § 3 above, and suppose further that X is compact Hausdorff and that every non-empty open subset of X has positive μ-measure. The ambiguous measure classes (defined at the end of § 3) form a finitely additive subalgebra $\mathcal{F}$ of $\mathcal{A}_m$. Put $X^* = R(\mathcal{F})$; the measure μ may then be regarded as defined on the open-closed subsets of X^*. It can be extended, in a standard way, to a countably additive measure μ^* on the Borel sets of X^*.

Theorem. There is a continuous surjection $\Theta_0 : X^* \to X$ that realizes an isomorphism between (X^*, μ^*) and (X, μ) .

In fact, one can define $\{\Theta_0(\alpha^*)\} = \bigcap \{F_1(a) : a \in \alpha^*\}$, where $F_1(a)$ is the smallest closed set in $a \in \mathcal{A}_m(X)$.

This theorem provides a relatively simple proof of a theorem of C. Ionescu Tulcea [15, p. 169]. Still assuming X compact Hausdorff, and that μ is positive for non-empty open sets, put $X' = R(\mathcal{A}_m(X))$. As before, the measure μ on X then gives a finitely additive measure on the open-closed subsets of X', and we extend this to a countably additive measure μ' on the Borel subsets of X'. The theorem in question asserts that (under the above hypotheses on X and μ) there is a continuous measure-preserving surjection $\Theta' : (X', \mu') \to (X, \mu)$. To see this, note that there is a natural continuous map $\phi : X' \to X^*$;

this follows from the fact that $X' = R(\mathcal{A}_m)$ and $X^* = R(\mathcal{F})$ where $\mathcal{F}$ is a (finitely additive) subalgebra of $\mathcal{A}_m$. Now take $\Theta' = \Theta_0 \circ \Phi$; it is not hard to verify that this works.

7. A completion for C(X)

We use $C(X)$, as usual, to denote the partially ordered linear space of all continuous real-valued functions on X. $C(X)$ is also, of course, a ring; but we are more concerned with its linear properties. Suppose that X is compact Hausdorff, and let Θ be the Gleason map from $\tilde{X} = R(\mathcal{A}_c(X))$ to X. Then Θ induces a linear-space isomorphism $\Theta^*: C(X) \to C(\tilde{X})$; and it is easy to see that Θ^* is also an order-isomorphism and a ring isomorphism. Now $C(\tilde{X})$, qua partially ordered set, is conditionally complete; that is, every bounded subset of $C(\tilde{X})$ has a least upper bound. (This follows from the fact that X is extremally disconnected.) Further, the image $\Theta^*(C(X))$ can be shown to be order-dense in $C(\tilde{X})$ (one first shows that if $f \in C(\tilde{X})$ is a characteristic function then there exists $g \in C(X)$ such that $0 \leq \Theta^*(g) \leq f$). Conversely, if $\psi : C(X) \to L$ is an arbitrary order-preserving linear-space isomorphism into a conditionally complete partially ordered linear space L, it can be shown that there is an order-preserving linear-space isomorphism $\Phi : C(\tilde{X}) \to L$ such that $\Phi \circ \Theta^* = \psi$. Thus, in a reasonable sense, $C(\tilde{X})$ <u>is the smallest conditionally complete partially ordered linear space containing</u> $C(X)$.

If X and Y are compact Hausdorff spaces, and $X \sim_c Y$, then $\tilde{X} = \tilde{Y}$, so that $C(X)$, $C(Y)$ will have the same "completions", in the above sense. It would be interesting to know whether the converse is true.

Essentially the same construction can be applied to all completely regular T_1 spaces X (not necessarily compact). We replace $C(X)$ by the subring $C_{ub}(X)$ of all uniformly continuous bounded functions (uniformly continuous in the uniformity induced by the finite open covers of X). Then $C_{ub}(X) = C(\beta X)$, and we apply the previous considerations to βX. Since $\mathcal{A}_r(X) = \mathcal{A}_r(\beta X) = \mathcal{A}_c(\beta X)$, the completion of $C_{ub}(X)$ will still be $C(\tilde{X})$ where now $\tilde{X} = R(\mathcal{A}_r(X))$.

Returning to the compact case, we note that the function space $C(\tilde{X})$ can be described more directly in terms of suitable classes of functions on X. Let $D(X)$ denote the set of all (real-valued) functions that are continuous and bounded (and defined) on residual subsets of X . Identify two functions in D(X) if they agree on a residual set. This produces a partially ordered linear space $\tilde{D}(X)$.

<u>Theorem. If X is compact Hausdorff, then</u> $C(\tilde{X}) = \tilde{D}(X)$, <u>to within a natural isomorphism.</u>

The isomorphism here is such that, for each $f \in C(X)$, the class $\tilde{f}$ (of f mod first category) in $\tilde{D}(X)$ corresponds to $g \in C(\tilde{X})$ where $g = f \circ \Theta$, Θ being the Gleason map. The proof depends on the fact that, because of the extremal disconnectedness of $R(\mathcal{A}_c(X))$, each real-valued function on X that is continuous when restricted to a residual set, is equal (mod first category) to one that is continuous on all of X .

It can be shown that, if X is compact and Hausdorff and satisfies the countable chain condition (i. e., has no uncountable family of pairwise disjoint open sets), then $\tilde{D}(X)$ is identical with the set of all bounded "analytically representable" functions, modulo sets of first category. (The analytically representable functions constitute the smallest family containing the continuous functions and closed under (pointwise) sequential limits.) The countable chain condition is not superfluous here, as is shown by essentially the same example as in § 3.

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