THE BOUNDARY OF THE SPECTRUM OF A LINEAR OPERATOR

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The results which I am communicating at this Symposium are selected from a joint paper by myself and one of my students, Dr. HERBERT A. GINDLER; this joint paper has been submitted for publication in *Studia Mathematica*. In this present Symposium report all proofs are omitted.

In all that I have to say X will be a complex normed linear space, not necessarily complete, and T will be a linear operator, not necessarily continuous or closed, with domain $D(T)$ and range $R(T)$ in X. I write $\lambda - T$ for $\lambda I - T$, where I is the identity mapping of X onto itself. All of the points λ in the complex plane are divided into two mutually exclusive and complementary sets: the resolvent set $\varrho(T)$ and the spectrum $\sigma(T)$. The boundary of the spectrum is denoted by $\partial\sigma(T)$. One may wish to make a finer classification of points by subdividing $\sigma(T)$ in some way. One such method of subdivision is well-known; the spectrum can be analyzed into point spectrum, continuous spectrum, and residual spectrum. Other schemes for subdividing the spectrum have been proposed. Any systematic study of the constituent parts of the spectrum, according to some scheme of subdivision, might be called a study of the *fine-structure* of the spectrum. I believe that such studies, if made along appropriate lines, will be of interest and importance for a better understanding of general spectral theory. I suppose the title of this report might just as well have been "Remarks and Results on Spectral Fine-Structure". Dr. Gindler and I do indeed have some results which belong to a general study of fine-structure, but they are far from complete. Our results on matters related to the boundary of the spectrum are also incomplete; I chose to emphasize the boundary in the title merely because we do have several quite explicit results pertaining to it.

I think it worth while to make a few general remarks about the idea of spectral fine-structure. Let α range over some index set, and let $\{P_\alpha\}$ be a family of properties which linear operators such as T may or may not possess. Suppose $\{P_\alpha\}$ is such that, if $\lambda - T$ has one of the properties P_α, then λ is in $\sigma(T)$, and if λ is in $\sigma(T)$ then $\lambda - T$ has a certain one of the properties P_α. Let $\sigma_\alpha(T)$ be the set of those λ such that $\lambda - T$ has property P_α. Let us further suppose that $\sigma_\alpha(T)$ and $\sigma_\beta(T)$ are disjoint if $\alpha \neq \beta$. Then the family of sets $\{\sigma_\alpha(T)\}$ provides us a decomposition of the spectrum, and thereby a possible basis for a study of spectral fine-structure. A part of the problem in general is to discover appropriate families of properties $\{P_\alpha\}$. These properties should be of interest and significance in the study of linear operators, and one should then seek to discover how these properties of operators are reflected in the nature of the sets of the family $\{\sigma_\alpha(T)\}$. For example, one may ask: For a given P_α, what limitations must be placed on a set of points in order that it may be, for a certain X and T, the set $\sigma_\alpha(T)$? Also, if one can prove that a certain set $\sigma_\alpha(T)$ is always open, this indicates that the property P_α of a linear operator is stable (i. e. it persists) when the operator is perturbed by adding to it an operator of the form cI, where $|c|$ is sufficiently small.

For our study of spectral fine-structure Dr. Gindler and I use nine properties of operators with domains and ranges in X, indicated by all the possible combinations in pairs of the symbols I, II, III, and 1, 2, 3. The first set of three symbols refers to the range of the operator:

I. The range is all of X.
II. The range is not all of X, but is dense in X.
III. The range is not dense in X.

The second set refers to the inverse of the operator:

1. The operator has a continuous inverse.
2. The operator has a discontinuous inverse.
3. The operator has no inverse.

Thus, for instance, an operator $\lambda - T$ has property I_3 if $R(\lambda - T) = X$ but $\lambda - T$ has no inverse. The corresponding part of $\sigma(T)$ is denoted by $I_3\ \sigma(T)$. This notation was originally introduced for the study of relations between an operator and its conjugate, by means of a "state diagram". (See the third and fourth items in the list of references.) With our notation, λ belongs to $\varrho(T)$ provided that $\lambda - T$ has one of the properties I_1, II_1, while λ belongs to $\sigma(T)$ if $\lambda - T$ has any of the other seven properties. We may note in passing that, if X is a complete space, a closed operator cannot possess either of the properties I_2, II_1. Hence, for a closed operator T and a complete space X, our subdivision of the spectrum is into six parts, corresponding to the properties

$$I_3, II_2, II_3, III_1, III_2, III_3.$$

As a simple but important tool we introduce the *minimum modulus* $\mu(T)$ of the operator T, defined as follows: $\mu(T) = \inf \|Tx\|$, the infimum being with respect to all unit vectors in $D(T)$. As is well known and easily proved, T has a continuous inverse if and only if $\mu(T) > 0$, and in that case $\mu(T)$ is the reciprocal of the norm of T^{-1}. For convenience let $\Phi(\lambda) = \mu(\lambda - T)$. We can easily show that Φ is continuous; therefore the set $\{\lambda : \Phi(\lambda) > 0\}$ is open. It consists of $\varrho(T)$ and $III_1\ \sigma(T)$. Both of these sets turn out to be open. Moreover, any connected subset of $\{\lambda : \Phi(\lambda) > 0\}$ either lies entirely in $\varrho(T)$ or entirely in $III_1\ \sigma(T)$. In particular, if $\Phi(\lambda_0) > 0$, the open disk $\{\lambda : |\lambda - \lambda_0| < < \Phi(\lambda_0)\}$ lies wholly in $\varrho(T)$ or wholly in $III_1\ \sigma(T)$. We see from the foregoing that we must have $\mu(\lambda - T) = 0$ whenever $\lambda \in \partial\sigma(T)$. However, one may construct examples where $\mu(\lambda - T) = 0$ at points λ not in $\partial\sigma(T)$.

Of the seven sets $I_2\ \sigma(T), \ldots, III_3\ \sigma(T)$ in our fine-structure decomposition of $\sigma(T)$, the only one of which it may be asserted that it is open, no matter how X and T

are chosen, is $\mathrm{III}_1\ \sigma(T)$. Examples exists to show that the other sets need not be open. As already noted, however, if X is complete and T is closed, $\mathrm{I}_2\ \sigma(T)$ is empty. If X is complete and T is closed, with domain dense in X, the set $\mathrm{I}_3\ \sigma(T)$ is open, for it may be shown in this case that $\mathrm{I}_3\ \sigma(T) = \mathrm{III}_1\ \sigma(T')$, where T' is the operator conjugate to T.

Suppose λ_0 is a point for which $\Phi(\lambda_0) > 0$. By the Φ-radius of λ_0 we mean the supremum of numbers $r > 0$ such that $\Phi(\lambda) > 0$ if $|\lambda - \lambda_0| < r$. Using the spectral mapping theorem, we have proved that when T is closed and X is complete, and λ_0 is in $\varrho(T)$, the Φ-radius of λ_0 is given by

$$\lim_{n\to\infty} \{\mu[(\lambda_0 - T)^n]\}^{1/n}. \tag{1}$$

This is the same as the distance from λ_0 to $\partial\sigma(T)$, if $\sigma(T)$ is not empty. If we drop the assumption that X is complete, and assume merely that $\Phi(\lambda_0) > 0$ (so that λ_0 may be either in $\mathrm{III}_1\ \sigma(T)$ or in $\varrho(T)$), we have the following less precise results: The Φ-radius of λ_0 is not less than

$$\sup_n \{\mu[(\lambda_0 - T)^n]\}^{1/n}, \tag{2}$$

provided either that (a) $D(T) = X$ and T is continuous, or (b) $\varrho(T)$ contains a point β such that $R(\beta - T) = X$. Still another set of conditions (c) leading to the same conclusion is the following: (c) $D(T^n)$ is dense in X for $n = 1, 2, \ldots$, and for each n the conjugate of $(\lambda_0 - T)^n$ is the same as the n-th power of the conjugate of $\lambda_0 - T$.

We have conjectured that, when $\Phi(\lambda_0) > 0$, the limit in (1) exists and coincides with the supremum in (2), but we have been unable to prove or disprove the conjecture. The following result can be proved: Suppose $D(T) = X$, T continuous, and $\Phi(\lambda_0) > 0$. Suppose there exists a continuous linear operator B which is an extension of $(\lambda_0 - T)^{-1}$ and such that $D(B) = X$. Finally, suppose $\limsup_{n\to\infty} \|(\lambda_0 - T)^n B^n\|^{1/n} \leqq 1$. Then the limit in (1) exists and is the reciprocal of the spectral radius of B. (We remark, incidentally, on the following useful facts, needed here and established by Dr. Gindler in his thesis: If A is a bounded linear operator defined on an incomplete space X, and if $\hat{A}$ is the unique continuous linear extension of A to the completion of X, then $\sigma(A) = \sigma(\hat{A})$, and the spectral radius of A is given, just as in the case of X complete, by $\lim_{n\to\infty} \|A^n\|^{1/n}$.)

H. A. Gindler and I have constructed many examples to show something of the variety which is possible in the fine-structure of $\sigma(T)$. In one interesting example, where $0 < a < b$, we have the following situation: $\sigma(T) = \{\lambda : |\lambda| \leqq b\}$, and the fine-structure is given by

$$\mathrm{III}_1\ \sigma(T) = \{\lambda : |\lambda| < a\}.$$
$$\mathrm{III}_2\ \sigma(T) = \{\lambda : a \leqq |\lambda| < \sqrt{(ab)}\}.$$
$$\mathrm{II}_2\ \sigma(T) = \{\lambda : \sqrt{(ab)} \leqq |\lambda| \leqq b\}.$$

We conclude this report with a rather curious result about $\partial\sigma(T)$ for the case in which T is a bounded linear operator in a complex Hilbert space X. For a bounded selfadjoint operator H we define

$$M(H) = \sup_{\|x\|=1} (Hx, x), \quad m(H) = \inf_{\|x\|=1} (Hx, x).$$

Evidently $m(-H) = -M(H)$ and $\mu(T) = \{m(T^*T)\}^{1/2}$. Now let Θ be real, and define

$$J_\Theta = \tfrac{1}{2}(e^{-i\Theta}T + e^{i\Theta}T^*).$$

We can then prove the following: If $\lambda = re^{i\Theta}$ (with $r \geqq 0$) is a point of $\partial\sigma(T)$, the polar coordinates (r, Θ) must satisfy each of the equations

$$r^2 = M[2rJ_\Theta - T^*T], \quad r^2 = M[2rJ_\Theta - TT^*].$$

References

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[4] *Angus E. Taylor*: Introduction to Functional Analysis. John Wiley, New York 1958.