DISTINGUISHED BOUNDARY SETS IN THE THEORY OF FUNCTIONS OF TWO COMPLEX VARIABLES

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1. Introduction. Every simply connected bounded domain of the z-plane can be conformally mapped onto the unit circle.

In analogy to mappings by functions of 1 c. v. (complex variable) in the theory of 2 (and more) c. v. we consider pseudo-conformal transformations, i. e. one-to-one mappings of the domains[1]) $\mathfrak{B}^4$ onto domain $\mathfrak{B}^{*4}$ by a pair

(1) $$z_k = z_k^*(z)\,, \quad k = 1, 2\,, \quad z = (z_1, z_2) \in \mathfrak{B}^4$$

of functions of 2 c. v. z_1, z_2; $z_k^*(z)$ are holomorphic in $\mathfrak{B}^4$. (The functional determinant $D(z_1, z_2) = [\partial(z_1^*, z_2^*)/\partial(z_1, z_2)]$ is finite and does not vanish in $\mathfrak{B}^4$.)

The problem to decide whether two given domains can be pseudo-conformally mapped onto each other, i. e., the classification of pseudo-conformally equivalent[2]) domains, is an important question of the theory of functions of 2 c. v.

Domains admitting the group of linear transformation $z_k^* = z_k \exp(i\varphi m_k)$, $0 \leqq \varphi \leqq 2\pi$, onto itself are called (m_1, m_2)-domains. Here m_k are integers, without a common factor.

A domain admitting the group $z_k^* = z_k \exp(i\varphi_k)$, $0 \leqq \varphi_k \leqq 2\pi$, $k = 1, 2$ onto itself is called a Reinhardt domain. These domains represent the simplest class of (m_1, m_2)-domains. Using the method of the kernel function we can decide whether the given bounded domain $\mathfrak{B}^4$ can be mapped pseudo-conformally on a Reinhardt circular domain $\mathfrak{R}^4$ and determine the mapping function of $\mathfrak{B}^4$ onto $\mathfrak{R}^4$.

The domains with a distinguished boundary represent another interesting subclass of domains.[3])

In the following we shall study some properties of the boundaries of domains with distinguished boundary sets. These investigations yield various invariants in the case of p.-c. transformations $\boldsymbol{T}$, regular in the closed domain (i.e., in the case of mappings (1) which satisfy the previously described requirements in $\overline{\mathfrak{B}^4}$).

Dinstinguished boundary sets are defined using either $\boldsymbol{C}(\mathfrak{B}^4)$ or $\boldsymbol{L}^2(\mathfrak{B}^4)$.

$\boldsymbol{C}(\mathfrak{B}^4)$ is the class of functions holomorphic in $\mathfrak{B}^4$ and continuous in $\overline{\mathfrak{B}^4}$.

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[1]) As a rule the upper index is the dimension of the set.

[2]) I. e. domains which by (1) can be mapped onto each other.

[3]) These two classes do not exclude each other. A bicylinder is a Reinhardt domain with a two-dimensional distinguished boundary.

$\mathbf{L}^2(\mathfrak{B}^4)$ is the class of functions $g(z)$ holomorphic in $\mathfrak{B}^4$ for which

$$\int_{\mathfrak{B}^4}|g(z)|^2 \, d\omega_z < \infty ,$$

$d\omega_z$ is the volume element, $\int$ is the Lebesgue integral.

In the first case we consider on the boundary $\boldsymbol{b}^3$ of $\mathfrak{B}^4$ *the smallest maximum set* $\mathfrak{D}$, i.e. the set of boundary points t, so that to every t, $t \in \mathfrak{D}$, a function $f(z) \in \mathbf{C}(\mathfrak{B}^4)$ exists with the property that[4])

$$|f(t)| > |f(z)| , \quad z \in \bar{\mathfrak{B}}^4 - t . \tag{2}$$

In the second case we determine the maximum of $|h(z)|^2$ of functions, which satisfy the conditions $\int_{\mathfrak{B}^4}|h(\zeta)|^2 \, d\omega_\zeta = 1$.

$$g(\zeta) = K(\zeta, \bar{z})/[K(z, \bar{z})]^{\frac{1}{2}} \tag{3}$$

is the function yielding this maximum and $|g(z)|^2 = K(z, \bar{z})$,[5]) $K \equiv K_{\mathfrak{B}^4}$, [5], p. 31 ff. For $z \to t$, where t is a boundary point of $\mathfrak{B}^4$, $K(z, \bar{z})$ goes (with some exceptions) to ∞. In a number of cases it has been shown that[6]) $\lim_{z \to t} (\widehat{zt})^n K(z, \bar{z})$ (for an appropriate n) does not vanish and is bounded. If $\boldsymbol{b}^3$ satisfies certain conditions, the order n equals 2, 3 or 4. See [6], chapt. I, p. 6 ff. Accordingly we introduce boundary points of n-th order, $n = 2, 3, 4$.

Remark. The geometrical structure of the boundary and certain invariants (derived from the kernel-function) exhibit different behaviour at boundary points of different order.

Let $\Phi(z)$ be holomorphic in $\bar{\mathfrak{B}}^4$ and $\Phi(z) = 0$ have only one point common with $\bar{\mathfrak{B}}^4$ ($\mathfrak{B}^4$ a bounded domain). If a function $\Psi(z)$ exists so that the p.-c. mapping

$$Z_1 = \Phi(z) , \quad Z_2 = \Psi(z) \tag{4}$$

is schlicht (one-to-one) in $\mathfrak{B}^4$, and the boundary $\boldsymbol{b}^3$ of $\mathfrak{B}^4$ at the point t is sufficiently regular, then t is of the third order. See [3]. If $\boldsymbol{b}^3 \cap [\Phi(z) = 0] = \mathfrak{S}^2$ is a (two-dimensional) segment of $[\Phi(z) = 0]$ and t is an interior point of $\mathfrak{S}^2$, then t is of the second order. If two analytic surfaces $\Phi(z) = 0$ and $\Psi(z) = 0$ pass through t and (4) is a schlicht mapping of $\mathfrak{B}^4$, then t is of the fourth order.

2. A generalization of the Schwarz lemma in domains with the smallest maximum boundary. To demonstrate the advantage of introducing the notion of the smallest maximum boundary in the present section, a generalization of the Schwarz lemma will be derived. Suppose that $\mathfrak{B}^4$ has a maximum boundary $\mathfrak{D}$ which is a proper part of $\boldsymbol{b}^3$. Further let us assume that $f(z)$, $p(z)$ and $f(z)/p(z)$ are functions of 2 c. v. which are holomorphic in $\bar{\mathfrak{B}}^4$. Finally let $p(z)$ vanish at least at one point of $\mathfrak{B}^4$, but $|p(\zeta)| > 0$ for $\zeta \in \mathfrak{D}$.[7]) Then

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[4]) It should be noted that the smallest maximum sets occur in the theory of rings. See [9].

[5]) In the following $\mathfrak{B}^4$ will be replaced by $\mathfrak{B}$, when it appears as a subscript.

[6]) If N is the interior normal at the point t, $(\widehat{zt})$ is the length of the projection on N of the segment connecting z and t. In the case of the points of the fourth order, $(\widehat{zt})$ is the distance between z and t.

[7]) If $p(z)$ vanishes at a point of $\mathfrak{B}^4$, $\min |p(Z)| = 0$ for $Z \in b^3$.

(1) $$|f(z)| \leqq |p(z)| \max_{\zeta \in \mathfrak{D}} |f(\zeta)| / \min_{\zeta \in \mathfrak{D}} |p(\zeta)|, \quad z \in \mathfrak{B}^4 .$$

See [4].

The inequality (1) can be improved as follows. Let $Q(\mathrm{Z})$, $\mathrm{Z} = (x_1, y_1, x_2, y_2)$, $z_k = x_k + iy_k$, be a family $\mathbf{Q}$ of B-harmonic functions which are regular in $\mathfrak{B}^4$. Further let $\mathbf{Q}_n$, $n = 1, 2, 3, \ldots$ be subclasses of[8]) $\mathbf{Q}$, $\mathbf{Q}_n \subset \mathbf{Q}_{n+1}$, $\lim_{n \to \infty} \mathbf{Q}_n = \mathbf{Q}$.

If we approximate $\log |p(\mathrm{Z})|$ in the Tchebysheff sense by the $Q_n(\mathrm{Z})$, i. e. so that

(2) $$\max_{z \in \mathfrak{D}} |\log |p(z)| - Q_n(\mathrm{Z})| = \min = m(n), \quad Q_n(\mathrm{Z}) \in \mathbf{Q}_n ,$$
$$z = (x_1 + iy_1, x_2 + iy_2), \quad \mathrm{Z} = (x_1, y_1, x_2, y_2),$$

then $m(n)$ is a non increasing function of n. Therefore

(3) $$\lim_{n \to \infty} m(n) = m(\infty)$$

exists. Since

(4) $$|Q_n(\mathrm{Z})| \leqq m(1) + \max_{z \in \mathfrak{D}} |\log |p(z)||$$

the $Q_n(\mathrm{Z})$ form a normal family in $\mathfrak{B}^4$. A subset $Q_{n_\nu}(\mathrm{Z})$ exists, so that $\lim_{\nu \to \infty} Q_{n_\nu}(\mathrm{Z}) = \mathbf{Q}^{(0)}(\mathrm{Z})$, $\mathrm{Z} \in \mathfrak{B}^4$. Thus

(5) $$|f(z)| \leqq |p(z)| \exp[-\mathbf{Q}^{(0)}(\mathrm{Z})] \max_{\mathrm{Z} \in \mathfrak{D}} |f(\mathrm{Z})| / e^{+m(\infty)} .$$

3. An analytic polyhedron. In order to gain a better insight in the theory of domains with distinguished boundary sets it is useful at first to restrict our considerations to the simplest type of these domains, namely to consider analytic polyhedra.

A domain bounded by finitely many segments of analytic hypersurfaces (see below) is called an *analytic polyhedron.*

We proceed to a precise description of domains to be considered in the following.

Let

(1) $$\mathfrak{H}^2(\lambda) = [\Phi(z, \lambda) = 0], \quad \lambda = \text{const}, \quad \Phi \text{ complex},$$

where $\lambda \in \mathfrak{s}^1 = [0 \leqq \lambda \leqq 1]$, $z = (z_1, z_2)$, be a family of surfaces. Here

(2) $$[\Phi(z, \lambda_1) = 0] \cap [\Phi(z, \lambda_2) = 0] = 0 \quad \text{for} \quad 0 \leqq \lambda_1 < \lambda_2 < 1$$

and

(3a) $$[\Phi(z, 0) = 0] \cap [\Phi(z, 1) = 0] = 0 \quad \text{(case I)}$$

or

(3b) $$[\Phi(z, 0) = 0] = [\Phi(z, 1) = 0] \quad \text{(case II)}$$

(Hyp. 1a). Here $\Phi(z, \lambda)$, $\lambda \in \mathfrak{s}^1$, $z \in \mathfrak{B}^4$, is a continuously differentiable function of the real variable λ and of the complex variables z_1, z_2. $\mathfrak{B}^4$ is a sufficiently large domain, see below (Hyp. 1b).

[8]) E. g. $\mathbf{Q}$ is the class of B-harmonic polynomials, while $\mathbf{Q}_n$ are real parts of polynomials $\Sigma a_{\nu\mu} z_1^\nu z_2^\mu$, $\nu + \mu \leqq n$ (B-harmonic = the real part of a holomorphic function of 2 c. v.).

Let

(4) $$\mathfrak{h}^3 = \bigcup_{\lambda=0}^{1} \mathfrak{H}^2(\lambda)\,.$$

$\mathfrak{M}^4$ is a domain bounded by segments $\mathfrak{j}_\kappa^3$ of hypersurfaces $\mathfrak{h}_\kappa^3$ introduced by (4). If $\mathfrak{m}^3$ is the boundary of $\mathfrak{M}^4$,

(5) $$\mathfrak{m}^3 = \bigcup_{\kappa=1}^{n} \mathfrak{j}_\kappa^3\,, \quad \mathfrak{j}_\kappa^3 = \mathfrak{h}_\kappa^3 \cap \overline{\mathfrak{M}}^4$$

(Hyp. 1c). The $\Phi_\kappa(z, \lambda_\kappa)$ are holomorphic functions of 2 c. v. in $\overline{\mathfrak{M}}^4$ (Hyp. 1d). Every $\mathfrak{J}_\kappa^2(\lambda_\kappa) = \mathfrak{H}_\kappa^2(\lambda_\kappa) \cap \mathfrak{m}^3$ can be uniformized (Hyp. 1e). This means that a pair of continuously differentiable functions $h_\kappa^k(Z_\kappa, \lambda_\kappa)$, $k = 1, 2$, $Z_\kappa \in \mathfrak{R}_\kappa^2(\lambda_\kappa)$, of the complex variable Z_κ, and of the real variable λ_κ, $\kappa = 1, 2, \ldots, n$, exist with the following property:

(6) $$\mathbf{R}_{\lambda_\kappa}: \qquad z_k = h_\kappa^k(Z_\kappa, \lambda_\kappa)\,, \quad k = 1, 2\,,$$

is a one-to-one mapping of the cylinder $\bigcup_{\lambda_\kappa \in \mathfrak{s}^1} \mathfrak{R}_\kappa^2(\lambda_\kappa)$ onto $\mathfrak{j}_\kappa^3$. (Hyp. 1f). ($\bigcup_{\lambda_\kappa \in \mathfrak{s}^1} \mathfrak{R}_\kappa^2(\lambda_\kappa)$ lies in the space λ_κ, Re Z_κ, Im Z_κ.) For every fixed λ_κ, the $h_\kappa^k(Z_\kappa, \lambda_\kappa)$ are functions of 1 c. v. Z_κ, holomorphic in $\mathfrak{R}_\kappa^2(\lambda_\kappa)$ (Hyp. 1g).

(7) $$\mathfrak{R}_\kappa^2(\lambda_\kappa) = [0 < \varrho_0 \leqq r_\kappa(\lambda_\kappa) < |Z_\kappa| < 1]$$

(Hyp. 1h).

Let $\Psi^{(k)}(z)$, $k = 1, 2$, be two holomorphic functions of 2 c. v. *in* a bounded domain $\overline{\mathfrak{K}}^4$. By the Weierstrass preparation theorem, $\mathfrak{K}^4$ can be covered by finitely many neighbourhoods $\mathfrak{N}_p^4$ so that in every $\mathfrak{N}_p^4$

(8) $$\Psi^{(k)}(z) = z_1^{\nu_{kp}} \prod_{\nu=1}^{n_{pk}} (z_1 - \varphi_{p\nu}^{(k)}(z_2))^{\mu_{pk}}\, \Omega_{pk}(z)\,, \quad k = 1, 2\,.$$

Here Ω_{kp} is a function which does not vanish in $\mathfrak{N}_p^4$, n_{pk} and μ_{pk} are integers.

If in every $\mathfrak{N}_p^4$

(9a) $$\nu_{1p} \cdot \nu_{2p} = 0\,,$$

(9b) $$\varphi_{p\nu_1}^{(1)}(z_2) \not\equiv \varphi_{p\nu_2}^{(2)}(z_2)\,,$$

$$\nu_1 = 1, 2, \ldots, n_{p1}\,, \quad \nu_2 = 1, 2, \ldots, n_{p2}\,,$$

then the $\Psi^{(k)}(z)$ will be called prime with each other in $\overline{\mathfrak{K}}^4$.

Lemma 3.1. *Let* $\overline{\mathfrak{j}}^3$ *be a given segment of an analytic hypersurface, which admits two representations*

(10) $$\overline{\mathfrak{j}}^3 = \bigcup_{\lambda \in \mathfrak{s}^1} \overline{\mathfrak{J}}^2(\lambda) = \bigcup_{\mu \in \mathfrak{s}^1} \overline{\mathfrak{H}}^2(\mu)$$

where $\overline{\mathfrak{J}}^2(\lambda) = [\Phi^{(1)}(z, \lambda)] = 0$, $\lambda \in \mathfrak{s}^1$ and $\overline{\mathfrak{H}}^2(\mu)$ $[\Phi^{(2)}(z, \mu) = 0]$, $\mu \in \mathfrak{s}^1$. *Further, for every* λ *and* μ *either* $\Phi^{(1)}(z, \lambda)$ *and* $\Phi^{(2)}(z, \mu)$ *are prime with each other in every sufficiently small domain* $\mathfrak{K}^4 \supset \overline{\mathfrak{j}}^3$ *or* $\overline{\mathfrak{J}}^2(\lambda)$ *is identical with* $\overline{\mathfrak{H}}^2(\mu)$. *Then*

(11) $$\overline{\mathfrak{J}}^2(\lambda) = \overline{\mathfrak{H}}^2(\mu^*(\lambda))$$

where $\mu^*(\lambda)$ *is a continuous and monotone function of* λ. *Compare* [7] *and* [8].

Therefore the decomposition of a segment $\mathfrak{j}^3$ into a sum of segments $\mathfrak{S}^2(\lambda)$ of analytic surfaces is essentially unique.

Let $\mathfrak{i}_\kappa^1(\lambda_\kappa)$ be the boundary curves of $\mathfrak{S}_\kappa^2(\lambda_\kappa)$,

$$\mathfrak{i}_\kappa^1(\lambda_\kappa) = \boldsymbol{R}_{\lambda_k}\{[|Z_\kappa| = r_\kappa(\lambda_\kappa)] \cup [|Z_\kappa| = 1]\} \tag{12}$$

and $\mathfrak{i}_{\kappa\vartheta}^1(\lambda_\kappa) = \mathfrak{i}_\kappa^1(\lambda_\kappa) \cap \mathfrak{S}_\vartheta^3$, $\vartheta \neq \kappa$,

$$\mathfrak{i}_\kappa^1 = \bigcup_{\substack{\vartheta=1 \\ \vartheta \neq \kappa}}^{n} \mathfrak{i}_{\kappa\vartheta}^1 , \tag{13}$$

$$\mathfrak{G}_{\kappa\vartheta}^2 = \bigcup_{\lambda_k \in \vartheta^1} \mathfrak{i}_\kappa^1(\lambda_\kappa) \cap \bigcup_{\lambda_\vartheta \in \vartheta^1} \mathfrak{i}_\vartheta^1(\lambda_\vartheta) , \tag{14}$$

$$\mathfrak{G}_\kappa^2 = \bigcup_{\substack{\vartheta=1 \\ \vartheta \neq \kappa}}^{n} \mathfrak{G}_{\kappa\vartheta}^2 . \tag{15}$$

Then

$$\mathfrak{D}^2 = \bigcup_{\kappa=1,\vartheta=1,\kappa\neq\vartheta}^{n} \mathfrak{G}_{\kappa\vartheta}^2 \tag{16}$$

is the distinguished boundary set of $\mathfrak{M}^4$. $\mathfrak{D}^2$ is a maximum boundary. Under some additional hypotheses, $\mathfrak{D}^2$ is the smallest maximum boundary.

4. Invariants in pseudo-conformal transformations of an analytic polyhedron. In accordance with the lemma 3.1 in a pseudo-conformal transformation $\boldsymbol{T}$ satisfying the condition (1.1) in $\mathfrak{M}^4$, the lamina $\mathfrak{S}_\kappa^2(\lambda_\kappa)$ is mapped onto a lamina $\mathfrak{S}_\kappa^{*2}(\lambda_\kappa)$ of the domain $\mathfrak{M}^{*4} = \boldsymbol{T}(\mathfrak{M}^4)$. Indeed, $\boldsymbol{T}^{-1}$ is an one-to-one transformation of $\mathfrak{S}_\kappa^{*2}(\lambda_\kappa)$ onto $\mathfrak{S}_\kappa^2(\lambda_\kappa)$, $\mathfrak{S}_\kappa^2(\lambda_\kappa)$ is a one-to-one image of $\mathfrak{R}_\kappa^2(\lambda_\kappa)$, therefore

$$\boldsymbol{P}_{\lambda_\kappa} : z_k^* = z_k^*[h_\kappa^1[Z_\kappa, \lambda_\kappa), h_\kappa^2(Z_\kappa, \lambda_\kappa)] , \quad k = 1, 2 , \tag{1}$$

is a one-to-one analytic mapping of $\mathfrak{R}_\kappa^2(\lambda_\kappa)$ onto $\mathfrak{S}_\kappa^{*2}(\lambda_\kappa)$. In every $\mathfrak{R}_\kappa^2(\lambda_\kappa)$ we consider the function

$$J_\kappa(Z_\kappa, \bar{Z}_\kappa, \lambda_\kappa) = \frac{\partial^2 \log K(Z_\kappa, \bar{Z}_\kappa)}{K\, \partial Z_\kappa\, \partial \bar{Z}_\kappa} , \quad K = K_{\mathfrak{R}_\kappa^2(\lambda_\kappa)} .$$

This function is invariant with respect to conformal transformations of $\mathfrak{R}_\kappa^2(\lambda_\kappa)$ onto itself. According to [12], [13], it assumes the value 2π on the boundary of $\mathfrak{R}_\kappa^2(\lambda_\kappa)$; at every interior point of $\mathfrak{R}_\kappa^2(\lambda_\kappa)$, $J_\kappa > 2\pi$. The function

$$B(z) = J_\kappa(Z_\kappa, \bar{Z}_\kappa, \lambda_\kappa) , \quad z = \boldsymbol{P}_{\lambda_\kappa}(Z_\kappa) , \tag{2}$$

is defined on the boundary $\mathfrak{m}^3$ of $\mathfrak{M}^4$. $B(z)$ is invariant with respect to pseudo-conformal transformations $\boldsymbol{T}$. At every point of the distinguished boundary $\mathfrak{D}^2$,

$$B(z) = 2\pi , \tag{3}$$

at every point $\mathfrak{m}^3 - \mathfrak{D}^2$,

$$B(z) > 2\pi . \tag{4}$$

In the case where $\mathfrak{R}^2_\kappa(\lambda_\kappa)$ are simply instead of doubly connected, the function $B(z)$ is constant on the whole boundary $\mathfrak{m}^3$, and therefore our procedure leads to a triviality. If $\mathfrak{R}^2_\kappa(\lambda_\kappa)$ is an n-ply connected domain, $n > 2$, we can construct the function $B(z)$ which has similar properties as in the case $n = 2$.

5. A second type of invariants. In § 4 we obtained in the case of analytic polyhedra invariants with respect to pseudo-conformal mappings $\boldsymbol{T}$. In the present section we shall consider invariants of a different type.

Now the hypothesis (1h) can be replaced by a weaker one, namely we assume here that $\mathfrak{R}^2_\kappa(\lambda_\kappa)$ is an n-ply connected domain $1 \leqq n \leqq N < \infty$.

The analytic polyhedron $\mathfrak{M}^4$ is obviously a complex. The $\mathfrak{S}^2_{\kappa\vartheta}$, see (3.16), are its edges. In a pseudo-conformal mapping $\boldsymbol{T}$, the boundary $\mathfrak{m}^3$ goes again into the boundary $\mathfrak{m}^{*3}$ of $\mathfrak{M}^{*4} = \boldsymbol{T}(\mathfrak{M}^4)$. The $\mathfrak{S}^{*2}_{\kappa\vartheta} = \boldsymbol{T}(\mathfrak{S}^2_{\kappa\vartheta})$. Since the mapping $\boldsymbol{T}$ is one-to-one and continuous the Betti group of the complex $\mathfrak{m}^3$ is preserved.

Similarly we form intersections of three (four) segments $\mathfrak{j}^3_{\kappa\vartheta}$ of analytic hypersurfaces, and we obtain a line (finite point set, respectively). Again the Betti groups of complexes obtained in this way are invariants with respect to pseudo-conformal transformations. This procedure can be extended to the case of general domains $\mathfrak{B}^4$ which possess the property that the kernel function $K_{\mathfrak{B}}$ is infinite of a certain order in the sense explained in § 1. Since the functional determinant of the mapping $\boldsymbol{T}$ does not vanish nor is infinite in $\mathfrak{B}^4$ the order of the infinity is preserved in this transformation, and thus the Betti group is an invariant with respect to the transformation $\boldsymbol{T}$.

6. Interior distinguished points. In § 5 we characterize the topological structure of the distinguished boundary. In analogy to this approach it is useful to apply topological methods in the study of the indicatrix of invariants at interior points of the domain.

$$(1) \qquad J(z) = \frac{K_{\mathfrak{B}}(z, \bar{z})}{T_{1\bar{1}}T_{2\bar{2}} - |T_{1\bar{2}}|^2}, \quad T_{m\bar{n}} = \frac{\partial^2 \log K(z, \bar{z})}{\partial z_m \partial \bar{z}_n}$$

is an invariant with respect to pseudo-conformal transformations. See [5] and [6]. The indicatrix[9]) $\mathfrak{n}^3(z_0)$ of an interior point z_0 of $\mathfrak{B}^4$ is divided by the hypersurface $J(z) = J(z_0)$ into parts. In a pseudo-conformal transformation the topological structure of the surface

$$(2) \qquad \mathfrak{T}^2(z_0) = \mathfrak{n}^3(z_0) \cap [J(z) = J(z_0)]$$

is preserved. Indeed, if the development of the functions (1.1) at $z_0 = (z_1^0, z_2^0)$ is

$$(3) \qquad (z_k^* - z_k^{*0}) = a_{10}^{(k)}(z_1 - z_1^0) + a_{01}^{(k)}(z_2 - z_2^0) + \dots,$$

the indicatrix will be transformed by

$$(4) \qquad Z_k = a_{10}^{(k)} Z_1 + a_{01}^{(k)} Z_2 .$$

[9]) $\mathfrak{n}^3(z_0) = \sum T_{m\bar{n}}(z_0)\xi_m\bar{\xi}_n = \varepsilon^2$, where $z_0 = (x_1^0 + iy_1^0, x_2^0 + iy_2^0)$ and $\varepsilon > 0$ is sufficiently small.

Since (1.1) is a one-to-one mapping, $0 < |a_{10}^{(1)}a_{01}^{(2)} - a_{10}^{(2)}a_{01}^{(1)}| < \infty$, and (4) is also a one-to-one and continuous mapping. Consequently it preserves the topological structure of $\mathfrak{T}^2(z_0)$. As a rule $\mathfrak{T}^2(z_0)$ is a sphere. ($J(z) = J(z_0)$ intersects $\mathfrak{n}^3(z_0)$ into two parts). Points z_0 where $\mathfrak{T}^2(z_0)$ is different from a sphere are called *interior distinguished points of the invariant* $J(z)$.

In the case of a Reinhard circular domain, the center is either an isolated maximum or minimum of $J_{\mathfrak{B}}(z)$ or a minimax and $\mathfrak{T}^2(z_0)$ is one or several tori. The center is the only point of this type.

A condition necessary for a domain $\mathfrak{B}^{*4}$ to be a pseudo-conformal image of a Reinhardt circular domain, is the existence of an interior distinguished point $z_0^* \in \mathfrak{B}^{*4}$. The point z_0^* is either an isolated maximum or minimum of $J(z)$, or the indicatrix $\mathfrak{T}^2(z_0^*)$ is one or the sum of several tori.

If one (and only one) interior distinguished point z_0^* (as described before) exists in $\mathfrak{B}^{*4}$, we determine the so-called representative domain $\mathfrak{R}^4(\mathfrak{B}^{*4}, z_0^*)$ with respect to z_0^*. See [2], [11]. If the domain $\mathfrak{B}^{*4}$ is pseudo-conformally equivalent to a Reinhardt circular domain, the obtained domain $\mathfrak{R}^4(\mathfrak{B}^{*4}, z_0^*)$ becomes a Reinhardt circular domain. Thus a method has been given to decide whether a given domain $\mathfrak{B}^{*4}$ can be mapped onto a Reinhardt circular domain or not. For details see [3], p. 48 and [10].

A similar method can be used to decide whether two domains, say[10]) $\mathfrak{B}^{*4}$ and $\mathfrak{B}^4$, can be pseudo-conformally mapped onto each other. We determine the interior distinguished points of $\mathfrak{B}^4$ ($\mathfrak{B}^{*4}$ respectivelly) which lie in the domain

$$J_1 \leqq J(z) \leqq J_2 \,. \tag{5}$$

(Here J_1 and J_2 are conveniently chosen constants.) The necessary conditions for $\mathfrak{B}^4$ to be pseudo-conformally equivalent to $\mathfrak{B}^{*4}$, is that

(a) the number n of the interior distinguished points $z^{(\nu)}$, and those of $z^{*(\mu)}$ in the domain (5) is the same;

(b) a correspondence between $z^{(\nu)}$ and $z^{*(\mu)}$ can be established so that in the corresponding points $z^{(\nu)}$ and $z^{*(\mu)}$, $J_{\mathfrak{B}}(z^{(\nu)})$ and $J_{\mathfrak{B}^*}(z^{*(\mu)})$ have the same value;

(c) the Betti group of $\mathfrak{T}^2(z^{(\nu)})$ and that of $\mathfrak{T}^2(z^{*(\mu)})$ in corresponding points are the same.

Under some additional assumptions, it is possible to show that n, $0 < n < \infty$, interior distinguished points exist in (5).

We construct the representative domains $\mathfrak{R}^4(\mathfrak{B}^4, z^{(\nu)})$ and $\mathfrak{R}^4(\mathfrak{B}^{*4}, z^{*(\mu)})$, $\nu = 1, 2, \ldots, n$, $\mu = 1, 2, \ldots, n$. The necessary and sufficient condition for $\mathfrak{B}^4$ to be pseudo-conformally equivalent to $\mathfrak{B}^{*4}$ is that domain $\mathfrak{R}^4(\mathfrak{B}^{*4}, z^{*(\mu)})$ can be obtained from $\mathfrak{R}^4(\mathfrak{B}^4, z(\nu))$ by a linear transformation [1], p 677 and [2].

[10]) We assume that $J_{\mathfrak{B}}(z)$ and $J_{\mathfrak{B}^*}(z)$ are not constant, and that there are isolated interior distinguished points of $J(z) \equiv J_{\mathfrak{B}}(z)$.

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