ON TWO-TO-ONE FUNCTIONS

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A function $f: X \to Y$ is said to be *two-to-one* if it is continuous and assumes every value in exactly two points. The space of arguments X is assumed to be metric and locally compact. In order to exclude the triviality we assume that Y is Hausdorff. It is known (Civin [1]) that such functions do not exist if X is an n-cell, where $n \leqq 3$ (the problem for $n > 3$ is open). The investigation of the two-to-one functions is in a natural manner equivalent to the investigation of an involution φ, where $\varphi(x)$ is the element of $f^{-1}f(x)$ different from x. This involution is, in general, discontinuous, but it is *semicontinuous*, i. e. for every $x \in X$ we have

$$\operatorname*{Ls}_{\xi \to x} \varphi(\xi) \subset x \cup \varphi(x) \cup p\,,$$

where p is the point adjoined to X by one-point compactification of X (here Ls denotes the topological limit superior in the sense of [2]). Civin showed that the investigation of φ on compact manifolds, or, if f is closed, on locally compact manifolds, is equivalent to the investigation of some continuous involution.

We do not assume that X is a manifold, or, if X is not compact, that f is closed. We consider the problem of behaviour of φ on neighbourhoods or on so called pseudoneighbourhoods of euclidean points or so called pseudoeuclidean points. According to this generality it is possible to obtain some results concerning the non-existence of two-to-one functions on some non locally connected continua (see [3]). We give some examples. One of them shows that there exist two-to-one functions on euclidean n-spaces for $n \geqq 2$ (the problem raised by Civin [1]).

1. *The general properties of involution* φ. Denote by $C(\varphi)$ the set of all continuity points of φ. It is an open and dense subset of X. The discontinuity point x of involution φ is said to be *weakly essential* (in short, x is a W-point of φ or $x \in W(\varphi)$) if $\Phi(x) = x \cup \varphi(x)$. It is said to be *strongly essential* (in short, x is an S-point of φ or $x \in S(\varphi)$) if $\Phi(x)$ contains p. A point $x \in X$ is said to be *pseudoeuclidean* if there exists a neighbourhood H of x in X such that the closure of the component of x in H is an euclidean solid sphere. We shall call such components H the euclidean *pseudoneighbourhoods.*

Theorem 1. *A pseudoeuclidean point* $x \in X$ *cannot be a* W*-point of* $\varphi \mid A$, *where* A *is the closure of an euclidean pseudoneighbourhood of* x *in* X.

In the proof we use a theorem of NEWMAN [4] concerning continuous involutions on closures of subdomains of compact manifolds and a theorem of KURATOWSKI [2], according to which, upper semicontinuous multi-valued functions are of the 1-st Baire class.

Theorem 2. *Let $R \subset X$ be a manifold such that for every $x \in R$ there exists a pseudoneighbourhood of x in X which is a neighbourhood of x in R simultaneously. If $\varphi \mid R$ has no S-points then the function*

$$\tilde{\varphi}(\xi) = \begin{cases} \varphi(\xi) & \text{for } \xi \in C(\varphi \mid R) \\ \xi & \text{for } \xi \in R - C(\varphi \mid R) \end{cases}$$

is continuous and one-to-one. If, in addition, $\tilde{\varphi}(R) \subset R$, then $\tilde{\varphi}$ is an involution on R and it cannot be the identity on open subsets of R.

2. *The case of locally compact manifolds.* According to Theorem 1, involution φ has no W-points if X is a manifold. However, if X is only a locally compact manifold then there can exist S-points. Consider the function $\tilde{\varphi} : X - S(\varphi) \to X$ defined by

$$\tilde{\varphi}(\xi) = \begin{cases} \varphi(\xi) & \text{for } \xi \in C(\varphi), \\ \xi & \text{for } \xi \in X - C(\varphi) - S(\varphi). \end{cases}$$

Theorem 3. *If X is a locally compact manifold without boundary then $\tilde{\varphi}$ is a continuous involution on $X - S(\varphi)$, and it cannot be the identity on open subsets of $X - S(\varphi)$.*

A homeomorphic image of the closed interval $0 \leqq t \leqq 1$, given by a homeomorphism h such that $h(0) = x$ and $h(t) \in X - S(\varphi)$ for $t \neq 0$, is said to be a *path* to the point x. A point $x \in S(\varphi)$ is said to be *strongly accessible* from $X - S(\varphi)$ if there exists a path to x such that $\lim_{t \to 0} \tilde{\varphi}\, h(t) = p$.

Theorem 4. *If X is a locally compact manifold, x is an S-point of φ, and U is an open neighbourhood of x in X, then there exist S-points of φ in U, being strongly accessible from $X - S(\varphi)$.*

The proof is similar to that of Theorem 1. Some corollaries are given in [3]. We quote here a simple one if X is the straight line, then φ has at most two S-points. From this, in an elementary way, we obtain that there do not exist two-to-one functions on the straight line.

3. *Examples.* Note first that it is possible to define two-to-one functions on some (infinite) dendrites. A more complicated example is an example of two-to-one function on a continuum being the closure of a plane simply connected domain, whose boundary is an irreductible cut of the plane into two domains (see for description [3]). This is in contrast to the non-existence of two-to-one functions on 2-cell. Both of these examples may be used in the proof that

Theorem 5. *There exist two-to-one functions on euclidean spaces E^n for $n \geqq 2$.*

The outline of construction is as follows. We consider E^n as $S^n - C$, where C is a continuum such that there exist two-to-one functions on it. Let f be one of them and let φ be the associated semicontinuous involution on C. Denote by C^* the image of C

by the antipodism on S^n, and assume that $C^* \cap C = 0$. In order to define two-to-one function on $S^n - C$, it is sufficient to define a suitable involution. This involution, λ, is given by

$$\lambda(x) = \begin{cases} x^* & \text{for } x \in S^n - C - C^*, \\ (\varphi(x^*))^* & \text{for } x \in C^*. \end{cases}$$

References

[1] *P. Civin*: Two-to-one mappings of manifolds. Duke Math. Journal *10* (1943), 49–57.

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[3] *J. Mioduszewski*: On two-to-one continuous functions. Rozprawy Matematyczne *24* (1961).

[4] *H. M. A. Newman*: A theorem on periodic transformations of spaces. The Quarterly Journal of Math., Oxford Series, *2* (1931), 1–8.