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ON PARACOMPACT SPACES AND RELATED QUESTIONS

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In § 1, the general notion of $\mathfrak{A}$-compactness of which the paracompactness is a special case is considered; the characterizations of such spaces are given, using systems of closed sets as well as using the notion of limit points of nets.

In § 2 it is shown that all paracompact spaces and only these spaces are limit spaces of simplicial projection spectra in the sense of P. ALEXANDROFF [1] (generalized by A. KUROSH [2]).

1. $\mathfrak{A}$-compact spaces. Let $\mathfrak{A} = \{\alpha\}$ be any system of open coverings of a given space X, containing all finite open coverings as subsystem.

We shall say that the space X is *$\mathfrak{A}$-compact* if each open covering of X has a refinement $\alpha \in \mathfrak{A}$. A system $\sigma = \{F\}$ of closed sets is called tangent to $\mathfrak{A}$, or simply *$\mathfrak{A}$-tangent*, if in each $\alpha \in \mathfrak{A}$ there is an element $V_\alpha \in \alpha$ intersecting all $F \in \sigma$. The following theorem is easily proved:

Theorem 1. *In order that a space X be $\mathfrak{A}$-compact it is necessary and sufficient that each $\mathfrak{A}$-tangent system has a non-void intersection.*

Obviously, the system of all closed sets containing a given point x is an $\mathfrak{A}$-tangent system which we shall denote by (x). If X is $\mathfrak{A}$-compact, (x) is a maximal $\mathfrak{A}$-tangent system and there are no maximal tangent systems other than those of the type (x). The correspondence $x \leftrightarrow (x)$ is a one-to-one correspondence between the points of the $\mathfrak{A}$-compact space X and the set Ξ of its maximal $\mathfrak{A}$-tangent systems. This correspondence becomes a homeomorphism if we introduce in Ξ a Wallman topology.

From now on we shall suppose that $\mathfrak{A} = \{\alpha\}$ is a directed system (with the natural ordering: $\alpha' \succ \alpha$ if the covering α' is a refinement of the covering α).

Take in any $\alpha \in \mathfrak{A}$ and a set $V_\alpha \in \alpha$. The system $\xi = \{V_\alpha\}$ thus obtained is directed by the directed system $\mathfrak{A} = \{\alpha\}$, this system ξ is called an *$\mathfrak{A}$-thread* if for any two $V_\alpha \in \xi$, $V_{\alpha'} \in \xi$, a $V_{\alpha''} \in \xi$ can be chosen with $\alpha'' \succ \alpha$, $\alpha'' \succ \alpha'$ (in $\mathfrak{A}$) and[1])

$$[V_{\alpha''}] \subseteq V_\alpha \cap V_{\alpha'} .$$

We shall say that the space X has the property $(K_{\mathfrak{A}})$ if for every $\mathfrak{A}$-tangent system $\sigma = \{F\}$ the sets $V_\alpha \in \alpha$ (having common points with all $F \in \sigma$) can be chosen in such a way as to form an $\mathfrak{A}$-thread ("the $\mathfrak{A}$-thread dual to the tangent system σ").

[1]) The brackets denote closure.

Theorem 2. *In order that a regular space X be $\mathfrak{A}$-compact it is necessary and sufficient that both of the following conditions are fulfilled:*

(a) *the space X has the property $K_{\mathfrak{A}}$,*

(b) *each $\mathfrak{A}$-thread $\xi = \{V_\alpha\}$ has a non-void intersection.*

It is natural to call a space $\mathfrak{A}$-*complete* if it satisfies the condition (b).

Lemma. *If $\xi = \{V_\alpha\}$ is an $\mathfrak{A}$-thread and $x \in \bigcap_\alpha [V_\alpha]$, then all of the neighbourhoods Ox of the point x are among the V_α.*

In fact, obviously $\bigcap_\alpha V_\alpha = \bigcap_\alpha [V_\alpha]$; for the given Ox we take a smaller O_1x with $[O_1x] \subseteq Ox$ and $\alpha_0 = \{Ox, X \setminus [O_1x]\}$; then necessarily $V_{\alpha_0} = Ox$.

It follows from this lemma that the intersection of all elements of a thread cannot contain more than one point.

Now the theorem 2 is proved in a few words. Let X be $\mathfrak{A}$-compact, and $\sigma = \{F\}$ an $\mathfrak{A}$-tangent system. Then $\bigcap_{F \in \sigma} F$ contains a point x_0.

In any α take an element $V_\alpha \ni x_0$. The system $\xi = \{V_\alpha\}$ thus obtained is an $\mathfrak{A}$-thread. In fact let $V_\alpha \in \xi$, $V_{\alpha'} \in \xi$ be given. Let us choose neighbourhoods Ox, O_1x of x so that

$$[Ox] \subseteq V_\alpha \cap V_{\alpha'}, \quad [O_1x] \subseteq Ox ,$$

and take the covering $\alpha_1 = \{Ox, X \setminus [O_1x]\}$. Take any covering α'' following α, α', α_1; then the set $V_{\alpha''} \in \xi$, containing x and contained in some element of α_1, must be contained in Ox; therefore

$$[V_{\alpha''}] \subseteq [Ox] \subseteq V_\alpha \cap V_{\alpha'} ;$$

q. e. d.

Obviously the thread ξ is dual to σ and the space X has the property $K_{\mathfrak{A}}$. Moreover, for any thread $\xi' = \{V'_\alpha\}$, the system $\{[V'_\alpha]\}$ is a tangent system and the necessity of our condition is proved.

Sufficiency: Let $\sigma = \{F\}$ be a tangent system and $\xi = \{V_\alpha\}$ a dual thread with

$$x_0 = \bigcap_\alpha V_\alpha = \bigcap [V_\alpha] .$$

As all V_α, i. e. all Ox_0, intersect all $F_\alpha \in \sigma$, we have $x_0 \in \bigcap_{F \in \sigma} F$ and thus X is $\mathfrak{A}$-compact.

Definition. A net $\{x_\vartheta\}$, $x_\vartheta \in X$, indexed by any directed set $\Theta = \{\vartheta\}$ is called an $\mathfrak{A}$-*net*, if every $\alpha \in \mathfrak{A}$ contains an element V_α such that for every $\vartheta_0 \in \Theta$ there is an $x_\vartheta \in V_\alpha$ with $\vartheta > \vartheta_0$.

Theorem 3. *In order that a regular space X be $\mathfrak{A}$-compact it is necessary and sufficient that it have the property $K_{\mathfrak{A}}$ and that each $\mathfrak{A}$-net have a limit point.*

Necessity: If X is $\mathfrak{A}$-compact, it has the property $K_{\mathfrak{A}}$. Let $\{x_\vartheta\}$ be an $\mathfrak{A}$-net. Let us define

$$F_\vartheta = [\mathscr{E}(x_{\vartheta'}, \vartheta' \geqq \vartheta)] .$$

Since $\{x_\vartheta\}$ is an $\mathfrak{A}$-net, $\{F_\vartheta\}$ is an $\mathfrak{A}$-tangent system, so that it has common point x_0 which is a limit point of $\{x_\vartheta\}$.

Sufficiency: Let $\sigma = \{F\}$ be an arbitrary $\mathfrak{A}$-tangent system, $\xi = \{V_\alpha\}$ a dual thread. For every α take $x_\alpha \in V_\alpha$; then $\{x_\alpha\}$ is a net (directed by $\mathfrak{A} = \{\alpha\}$), and in fact an $\mathfrak{A}$-net. By hypothesis, it has a limit point x_0 which is the (only) common point of all $[V_\alpha]$. Thus by the above lemma, all neighbourhoods of x are among the V_α, so that x belongs to all $F \in \sigma$ and $\bigcap_{F \in \sigma} F \neq \emptyset$, q. e. d.

2. Paracompactness, metric and projective spectra. First of all we recall the following theorem, proved (but not formulated explicitly) by C. H. Dowker (1948); an explicit formulation can be found in M. Katětov's Appendix to the book „Topologické prostory" (Topological spaces, Prague 1959) by E. Čech.

Theorem 4. *In order that a regular space X be paracompact it is necessary and sufficient that for every open covering ω of X there exist an ω-mapping*[2]) *of X onto a metric space Y. If we suppose that Y is metric separable, we obtain a characterisation of final compact (Lindelöf) spaces.*

The proof of the first part of this theorem is straight-forward: if X allows, for every ω, an ω-mapping onto a paracompact space Y, then X itself is paracompact.

The proof of the second part is contained in a result of C. H. Dowker [3]. An alternate proof is given in the book mentioned above.

Now let us pass to the spectral characterization of paracompact spaces.

1. According to a classical definition of P. Alexandroff, a projection-spectrum is a directed set Σ of simplicial complexes[3]) $\alpha, \alpha', \ldots$ and of simplicial mappings, called projections; for each pair α, α' in Σ with $\alpha' \succ \alpha$ there is a well defined projection $\pi_\alpha^{\alpha'}$ of the complex α' onto α; for $\alpha'' \succ \alpha' \succ \alpha$ one has

$$\pi_\alpha^{\alpha''} = \pi_\alpha^{\alpha'} \pi_{\alpha'}^{\alpha''} .$$

If in each complex α we take a simplex t_α under the condition

$$\pi_\alpha^{\alpha'} t_{\alpha'} = t_\alpha ,$$

we obtain a so-called thread $\xi = \{t_\alpha\}$ of the spectrum; a thread $\xi = \{t_\alpha\}$ is called maximal if there exists no thread $\xi' = \{t'_\alpha\}$ different from ξ and such that $t'_\alpha \geqq t_\alpha$ (that is to say that t_α is a face of t'_α) for all t_α.

By definition, the maximal threads are points of the limit space $\tilde{\Sigma}$ of the spectrum

$$\Sigma = \{\alpha, \pi_\alpha^{\alpha'}\} .$$

As for the topology of $\tilde{\Sigma}$, we define for any simplex t_{α_0} of a given $\alpha_0 \in \Sigma$ the set Ot_{α_0} consisting of all threads $\xi' = \{t'_\alpha\}$ with $t'_{\alpha_0} \leqq t_{\alpha_0}$. These sets Ot_α are by definition the basic open sets of $\tilde{\Sigma}$. It is easily seen that the set $\tilde{\Sigma}$ with this topology is a T_1-space.

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[2]) Let ω be a covering of the space X; a continuous mapping $f: X \to Y$ is called an ω-mapping (Alexandroff [1]), if each point $y \in Y$ has a neighbourhood Oy such that $f^{-1}Oy$ is contained in some element of ω.

[3]) A complex is meant in the classical sense, as a set α of (abstract finite dimensional) simplices; if $t \in \alpha$ and $t' < t$ (i. e. t' is a face of t), then $t' \in \alpha$.

Now let us consider for any simplex $t_{\alpha_0} \in \alpha$ the set Φt_{α_0} of all points

$$\xi' = \{t'_\alpha\} \in \tilde{\Sigma} \quad \text{with} \quad t'_{\alpha_0} \geqq t_{\alpha_0} .$$

It is easily proved that the sets Φt_α are closed in the topological space $\tilde{\Sigma}$. Among the Φt_{α_0}, the sets Φe_α corresponding to the vertices e_α of the complex α are the most important.

For a given complex $\alpha \in \Sigma$, the sets Φe_α corresponding to all vertices of α form a closed covering φ_α of the space $\tilde{\Sigma}$.

These coverings φ_α are called the *fundamental coverings* of the limit space $\tilde{\Sigma}$.

Remark 1. One proves immediately that the *nerve of the covering φ_α is a subcomplex of the complex α.*

Now call the spectrum Σ *complete* if for every $t_{\alpha_0} \in \alpha_0 \in \Sigma$ there exists a thread $\xi = \{\tau_\alpha\}$ with $\tau_{\alpha_0} \geqq t_{\alpha_0}$. If the spectrum

$$\Sigma = \{\alpha, \pi_\alpha^{\alpha'}\}$$

is complete, then the nerve of φ_α is the complex α.

Remark 2. It is easy to give a condition for the regularity of the limit space $\tilde{\Sigma}$ of the spectrum

$$\Sigma = \{\alpha, \pi_\alpha^{\alpha'}\} .$$

Call Σ a regular spectrum if for any

$$\xi = \{\tau_\alpha\} \in \tilde{\Sigma}$$

and α_0 there exists an $\alpha' \in \Sigma$ such that supposing

$$\tau_{\alpha'} = |e_{\alpha'}^0, \ldots, e_{\alpha'}^2| \in \xi ,$$

we have

$$\Phi e_{\alpha'}^0 \cup \ldots \cup \Phi e_{\alpha'}^r \subseteq O\tau_\alpha .$$

A regular spectrum has a regular limit space.

2. All the previous notions are either those described in the classical paper [1] of Alexandroff, in which the definition of a projective spectrum is given, or their immediate generalizations. Now we come to the main condition, which expresses that the convergence of the spectrum to its limit space is in a certain sense uniform.

Definition. The spectrum $\Sigma = \{\alpha, \pi_\alpha^{\alpha'}\}$ is called *uniform* if any covering of Σ by basic open sets is refined by some fundamental covering φ_α.

The principal result of this paper is:

Theorem 5. *The limit space of any uniform (regular) spectrum is a paracompact (regular) space.*

Every paracompact regular space is the limit space of a uniform regular complete spectrum.

The strong paracompact spaces[4]) *are characterized among paracompact*

[4]) Strong paracompact means that any covering can be refined by a star-finite one.

spaces by the condition that all complexes in the spectrum can be supposed star finite.

Let us say only a few words about the proof of the second part of this theorem.

If X is a paracompact (regular and therefore normal) space, then every open covering ω of X can be refined by a locally finite canonical (closed) covering.[5]) These coverings form a directed system. Their nerves (star-finite if the covering is star-finite) with the natural projections form a uniform regular complete spectrum Σ with the limit space $\tilde{\Sigma}$ homeomorphic to X.

Finally, let us remark that for a spectrum $\Sigma = \{\alpha, \pi_{\alpha}^{\alpha'}\}$ composed of finite complexes (that is the classical case of Alexandroff-Kurosch with a bicompact limit space), the condition of uniformity fundamental in our theorem is satisfied automatically.

References

[1] *P. Alexandroff*: Gestalt und Lage abgeschlossener Mengen beliebiger Dimension. Annals of Math. *30*, 1929, 101—187.

[2] *A. Kurosch*: Kombinatorischer Aufbau der bikompakten topologischen Räume. Comp. Math. *2* (1935), 471—476.

[3] *C. H. Dowker*: On Alexandroff's Mapping Theorem. Bull. Amer. Math. Soc. *54*, 4, 1948, 386—391.

[5]) A covering is canonic if its elements are closures of disjoint open sets.