RELATIONS ON TOPOLOGICAL SPACES

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The subject which I shall discuss has not long been investigated by mathematicians in this abstract form. The earliest papers on this subject seem to be those of C. PAUC [10] and S. EILENBERG [2] to which I shall return. I first used some of these notions in a small paper on fixed points in 1945 [12]. Later, in 1950, L. NACHBIN [9] published a small book on some aspects of this theory and since then it has been the subject of investigation by some of my students, Professors R. J. KOCH, I. S. KRULE, and L. E. WARD, JR., as well as by myself.

Let me first consider, to introduce the subject, a very old result: *A continuous real function on a closed and bounded interval attains, at some point, its minimum.*

Instead of a closed and bounded interval, let us take for the domain of the function a compact Hausdorff space (non-void) and denote by

$$f: X \to R$$

a real continuous function. We define a set P in $X \times X$ (the cartesian product of the space X with itself) by

$$P = \{(x, y) \mid f(x) \leqq f(y)\} .$$

We put the old notation $\leqq$ to a new use and write, for $x, y \in X$,

$$x \leqq y \leftrightarrow (x, y) \in P .$$

Let us say of any subset of $X \times X$ that it is a relation on X. Then clearly, the relation P is

reflexive:

$$(x, x) \in P \quad \text{for any} \quad x \in X ,$$

transitive:

$$(x, y) \in P \quad \text{and} \quad (y, z) \in P \to (x, z) \in P .$$

It is also (since f is continuous) a closed subset of $X \times X$, where we use the standard cartesian product topology.

If we could exhibit an element a of X such that, in the new sense,

$$x \leqq a \to a \leqq x \quad (a \text{ is } P\text{-minimal}) ,$$

whatever be x in X, then we would have a point of X at which our function f attains its minimum. We have, indeed, the following result which is sufficient for our purpose though more inclusive results are known:

Theorem I. *If X is a compact Hausdorff space and if P is a non-void closed transitive relation on X then there is at least one P-minimal element.*

The proof of this, as one would expect, is not difficult and we indicate the train of the reasoning. A P-chain (for any relation P on any space X) is such a set $C \subset X$ that $x, y \in C$ implies $(x, y) \in P$ or $(y, x) \in P$. In virtue of the very well-known Hausdorff Maximality principle (Zorn's Lemma), there is for any relation on any space a maximal P-chain.

Now, for $x \in X$, let Px be the set of all $y \in X$ such that $(y, x) \in P$.

Since P is closed it follows readily that Px is closed for any $x \in X$. If C is a maximal P-chain, then

$$\bigcap\{Px \mid x \in C\}$$

is non-void since X is compact and P is non-void and any one of its elements is P-minimal.

The theorem appears in the second edition of Birkhoff's book on lattice theory though it is implicit in my paper of 1945 and was certainly to be obtained by any mathematician who was interested in these matters. Strangely, except for this, Birkhoff's book does not contain any other results of this sort.

The general question which is of interest to me is this: *What spaces admit what sort of relations.*

Now we know that any compact Hausdorff space X can be imbedded in a Tychonoff cube T (in many ways) and we may define, assuming actually that $X \subset T$,

$$x \leqq y \leftrightarrow x_t \leqq y_t \text{ for each coordinate index } t\,.$$

The so-defined relation is a closed partial order (reflexive, antisymmetric and transitive). But generally, such a relation cannot be a total order and need not have other interesting properties as we shall now see.

Let us say of a point p of a connected space X that it is a *cutpoint* if $X \setminus p$ is the union of two non-void disjoint open sets.

An *arc* is a compact connected space containing more than one point and such that every point (save at most two) is a cutpoint. A real arc is an arc that contains a countable dense set (is separable). A real arc is homeomorphic with the closed unit interval.

A relation P is *total* if for every (x, y) we have either $(x, y) \in P$ or $(y, x) \in P$, which is to say that X is a chain. A *partial order* is a reflexive, antisymmetric and transitive relation and a *total order* is a partial order that is also total.

Theorem 2. *A compact connected Hausdorff space admits a closed total order if and only if it is an arc. A connected Hausdorff space admits a closed order if and only if the complement of the diagonal in $X \times X$ is not connected.* (See C. Pauc [10], S. Eilenberg [2], and R. L. Wilder [18].)

It is a very well known result of R. L. Moore (see [18]) that if a space is compact connected locally connected and metrizable then any two points of it are endpoints of a real arc. The question of whether or not "metrizable" could be replaced by "Hausdorff" (deleting "real") remained open for some years and was finally solved by Mardešić [8] in the negative.

However, there is a very useful and elegant counter-theorem due to R. J. Koch [4].

Theorem 3. *Suppose that X is a compact Hausdorff space, that P is a closed partial order on X and that W is an open set in X. If for each $x \in W$ each open set about x contains an element y with $(y, x) \in P \setminus (x, x)$ then every element a of W belongs to a compact connected P-chain which contains at least one point not in W and of which a is the maximal element.*

Of course, the chain in the theorem is an arc.

Let us agree that a *continuum* is a compact connected Hausdorff space and let us say that a continuum is *indecomposable* if it is not the union of two of its proper subcontinua. The existence of indecomposable continua was shown by L. E. J. Brouwer and they have since been the subject of many papers.

We say that P is left monotone if Px is connected (and hence a continuum) for each $x \in X$, recalling that

$$Px = \{y \mid (y, x) \in P\}.$$

It is suggestive to write, as earlier,

$$x \leqq y \leftrightarrow (x, y) \in P$$

so that

$$Px = \{y \mid y \leqq x\}.$$

For $A \subset X$ it is convenient to write

$$PA = \bigcup\{Px \mid x \in A\}$$

and say that P is continuous if $PA^* \subset (PA)^*$ for each $A \subset X$, where * denotes closure.

Theorem 4. ([13]). *There exists no continuous closed left monotone partial order on an indecomposable continuum, other than the identity.*

A function on X to X is a special sort of relation on X and the notation we have introduced is designed to reinforce this. Here is a "fixed point theorem":

Theorem 5. *If X is a continuum, if P is a closed continuous left monotone partial order on X and if z separates Pa and Pb in X, then $Pz = z$.*

Theorem 5 may be used to prove:

Theorem 6. *If X is a continuum and if P is a closed continuous left monotone partial order on X then the set of minimal elements is connected.*

Proof. We omit the proof that the set of minimal elements is closed and suppose that this set is the union of the disjoint non-void closed sets A and B. Let us write

$$P^{(-1)}A = \{x \mid Px \cap A \neq \square\}.$$

In virtue of Theorem 1 it may be seen that

$$X = P^{(-1)}A \cup P^{(-1)}B \text{ and } P^{(-1)}A \cap P^{(-1)}B \neq \square.$$

There exists, in virtue of the Hausdorff Maximality Principle, a maximal P-chain C in the set $P^{(-1)}A \cap P^{(-1)}B$. If z is the minimal element of C then Pz is a continuum with z as cutpoint and, indeed, z separates some element of A from some element of B in Pc. Thus $Pz = z$ and so z is a minimal element which is not in either A or B, a contradiction.

There are other "fixed point theorems", that are not unrelated to the above, [1] and [3].

It is interesting to have criteria which establish the existence of universally maximal and minimal elements. The hypotheses for such propositions are, of course, necessarily quite strong. Here is an example:

Theorem 7. *If P is a left monotone closed partial order on the 2-sphere such that the set of P-minimal elements is connected and such that no set Px cuts the 2-sphere then there is a universally P-maximal element.*

Characterization of certain spaces, not unrelated to those given above, have been found by various authors — R. J. Koch-I. S. Krule [5], I. S. Krule [7] and L. E. Ward [17]. But for lack of time it is not possible to say any more about this interesting problem.

The bibliography is indicative rather than definitive and it has been selected so that the bibliographies of the papers listed form a rather complete account of what has been published in this area.

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