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A contribution to the descriptive theory of sets and spaces

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# A CONTRIBUTION TO THE DESCRIPTIVE THEORY OF SETS AND SPACES

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Praha

In the present note the following kinds of spaces are investigated: bianalytic spaces (Baire sets of compact spaces; for the definition of Baire sets see 1.10 and 1.5), Borelian spaces (one-to-one continuous images of bianalytic spaces), analytic spaces (continuous images of bianalytic spaces), one-to-one continuous images of closed subspaces of the space of all irrational numbers, and continuous images of the space of all irr. numbers. As an introduction one can make use of [6] and [9]. For historical notes see Section 5. It should be noted that the notation, terminology and results of Section 1 are used without references, the proofs of most results of Section 3 do not depend upon Section 2, and Section 4 essentially depends upon the preceding sections. The proofs of Section 2 are relatively brief because they are similar to those of [6].

For convenience, all spaces under consideration are supposed to be completely regular.

## 1. NOTATION AND TERMINOLOGY

**1.1.** exp $H$ always denotes the family of all subsets of the set $H$.

**1.2.** Let $f$ be a mapping of $H$ onto $L$. If $\mathcal{M} \subset \exp H$, then $f[\mathcal{M}]$ denotes the family of all $f[M]$, $M \in \mathcal{M}$. If $\mathcal{M} \subset \exp L$, then $f^{-1}[\mathcal{M}]$ denotes the family of all $f^{-1}[M]$, $M \in \mathcal{M}$.

**1.3.** If $\mathcal{M} \subset \exp H$ and $N \subset H$, then $\mathcal{M} \cap N$ $(= N \cap \mathcal{M})$ denotes the family of all $M \cap N$, $M \in \mathcal{M}$.

**1.4.** A centered family of sets is a family $\mathcal{M}$ with the finite intersection property, i.e. the intersection of any finite subfamily of $\mathcal{M}$ is non-void.

**1.5.** For any family $\mathcal{M} \subset \exp H$, $\mathcal{B}^*(\mathcal{M})$, $\mathcal{B}(\mathcal{M})$, $\mathcal{B}_*(\mathcal{M})$ denote, respectively, the smallest families containing $\mathcal{M}$ and closed under following operations:

(a) countable unions and complementation,

(b) countable unions and countable intersections,

(c) countable intersections and countable disjoint unions.

**1.6.** By a complemented part of a family of sets $\mathcal{M}$ is meant the following family

$$\text{compl. p. } \mathcal{M} = \{M : M \in \mathcal{M}, (L - M) \in \mathcal{M}\} ,$$

where $L$ is the union of $\mathcal{M}$. An $\mathcal{M}$ will be called complemented if

$$\text{compl. p. } \mathcal{M} = \mathcal{M} .$$

**1.7.** The letters $S$ and $\Sigma$ will always be used to denote the set of all finite sequences and infinite sequences, respectively, of positive integers. $S_n$, $n = 1, 2, \ldots$, denotes the set of all $s \in S$ of length $n$. We shall write $s \prec \sigma$ if $s$ is a section of $\sigma$, i. e., if $s = \{s_1, \ldots, s_n\}$ and $\sigma = \{\sigma_i\}$, then $s_i = \sigma_i$ for all $i \leqq n$.

**1.8.** A determining system[1]) in a family $\mathcal{M}$ of sets is a mapping $M = \{M(s)\}$ of $S$ to $\mathcal{M}$ such that

$$M(\{s_1, \ldots, s_n, k\}) \subset M(\{s_1, \ldots, s_n\}) . \tag{1}$$

The nucleus of the determining system $M$ is the set

$$\mathscr{A}(M) = \mathscr{A}(\{M(s)\}) = \bigcup_{\sigma \in \Sigma} \bigcap_{s < \sigma} M(s) . \tag{2}$$

The nuclei of determining systems in $\mathcal{M}$ will be called $\mathcal{M}$-Souslin sets or Souslin with respect to $\mathcal{M}$. The family of all $\mathcal{M}$-Souslin sets will be denoted by $\mathscr{A}(\mathcal{M})$. It is well known that

$$\mathscr{A}(\mathscr{A}(\mathcal{M})) = \mathscr{A}(\mathcal{M}) , \tag{3}$$

that means, the family of all $\mathcal{M}$-Souslin sets is closed under the Souslin operation. Of course, by the Souslin operation is meant the operation leading from $M$ to $\mathscr{A}(M)$.

**1.9.** By a space is meant a completely regular topological space. The letters $T$, $X$, $Y$, $Z$ always denote spaces. For any $X$, $\mathsf{F}(X)$, $\mathsf{Z}(X)$ and $\mathbf{K}(X)$ denote, respectively, the family of all closed sets of $X$, zero-sets of $X$ and compact sets contained in $X$. Of course, zero-sets of $X$ are sets of the form $f^{-1}(0)$, where $f$ is a real-valued continuous function on $X$. The closure in $X$ of $M \subset X$ will be denoted by $\overline{M}^X$ or merely $\overline{M}$. If $\mathcal{M} \subset \exp X$, then $\overline{\mathcal{M}}^X$ or merely $\overline{\mathcal{M}}$, will be used to denote the family of all $\overline{M}^X$, $M \in \mathcal{M}$.

**1.10.** The elements of $\mathscr{B}^*(\mathsf{F}(X))$ will be called Borel subsets of $X$. According to M. Katětov [13], the elements of $\mathscr{B}^*(\mathsf{Z}(X)$ will be called the Baire sets of $X$. The family $\mathscr{B}(\mathsf{Z}(X)$ is complemented, for complements of zero-sets are countable unions of zero-sets. Thus

$$\mathscr{B}(\mathsf{Z}(X)) = \mathscr{B}^*(\mathsf{Z}(X)) . \tag{4}$$

If $X$ is metrizable, or more generally, perfectly normal, then $\mathsf{Z}(X) = \mathsf{F}(X)$ and hence

$$\mathscr{B}^*(\mathsf{F}(X)) = \mathscr{B}(\mathsf{Z}(X)) . \tag{5}$$

To my knowledge, it is not known whether (5) implies that $X$ is perfectly normal (this is a problem of M. Katětov [13]).

**1.11.** By a perfect mapping of $X$ onto $Y$ is meant a continuous and closed mapping of $X$ onto $Y$ such that the preimages of points are compact. If $f$ is a perfect map-

[1]) By a determining system in $\mathcal{M}$ is usually meant a mapping of $S$ to $\mathcal{M}$ and the determining system satisfying (1) is called regular.

ping of $X$ onto $Y$, and $M \subset X$ is either closed in $X$ or $f^{-1}[f[M]] = M$, then the restriction of $f$ to $M$ is a perfect mapping.

**1.12.** A class **D** of spaces will be called F-hereditary if closed subspaces of spaces from **D** belong to **D**. **D** will be called productive (countably productive, finitely productive) if the topological product of any (countable, finite) indexed family of spaces from **D** belongs to **D**.

**1.13.** Let $\mu = \{\mathcal{M}\}$ be a collection of coverings of $X$. An $\mu$-Cauchy family is a centered family $\mathcal{N}$ of subsets of $X$ such that $\mathcal{N} \cap \mathcal{M} \neq \emptyset$ for all $\mathcal{M}$ in $\mu$. The collection $\mu$ is said to be *complete* if the intersection of $\overline{\mathcal{N}}$ is non-void for every $\mu$-Cauchy family $\mathcal{N}$.

One can prove the following results (the proofs may be found in [7]): Let $f$ be a mapping of $X$ onto $Y$. If $f$ is perfect and $\mu$ is a complete collection in $Y$, then $f^{-1}[\mu] = \{f^{-1}[\mathcal{M}]\}$ is complete in $X$. If $f$ is continuous and one-to-one and $\mu$ is complete in $X$, then $f[\mu] = \{f[\mathcal{M}]\}$ is complete in $Y$.

## 2. ANALYTIC AND BIANALYTIC SPACES

In the classical descriptive theory of sets, and also in that presented here, the space of irrational numbers plays an important rôle. It is well known that the space of all irrational numbers is homeomorphic with the product space $\Sigma = N^N$, where $N$ is the discrete space of positive integers. Thus $\Sigma$ is the set of all infinite sequences of positive integers with the topology of pointwise convergence. Setting

$$f(\sigma) = \frac{1}{\sigma_1 + 1/(\sigma_2 + \ldots)}$$

we obtain a homeomorphism of $\Sigma$ onto the space of all irrational numbers of the interval $\langle 0, 1 \rangle$.

In the classical theory, continuous images of $\Sigma$, if metrizable, are called analytic sets. G. Choquet first showed that continuous images of spaces belonging to $\mathcal{B}(\mathbf{K}(X))$ for some $X$ ($K$-analytic spaces in his terminology), if metrizable, are analytic sets.

In [6] and [8] an internal characterisation of $K$-analytic (analytic in our terminology) is given. Here we make use of another definition, formally similar to that of analytic sets.

**Definition 1.** A space $X$ will be called *analytic* if there exists a continuous compact multi-valued mapping of $\Sigma$ onto $X$. The class of all analytic spaces will be denoted by **A** and the family of all $A \subset X$, $A \in \mathbf{A}$, by $\mathbf{A}(X)$.

By a multivalued mapping of $X$ onto $Y$ is meant a mapping $f$ of $X$ into exp $Y$ such that the union of all $f(x)$ is $Y$. A mapping $f$ will be called continuous, if for any $x \in X$ and any open set $U \supset f(x)$ there exists a neighborhood $V$ of $x$ with

$$y \in V \Rightarrow f(y) \subset U\,.$$

Setting

$$f^{-1}[M] = \{x : x \in X, f(x) \subset M\},$$

we can say that $f$ is continuous if and only if $f^{-1}[U]$ is open for any open $U$. The composition $h = g \circ f$ of two multi-valued mappings $f$ and $g$ is defined as follows:

$$h(x) = \bigcup \{g(y) : y \in f(x)\}.$$

Clearly the composition of two continuous multi-valued mappings is continuous. $f$ will be called compact, if all sets $f(x)$ are compact. It is easy to prove that the image of a compact (resp. Lindelöf) space under a continuous compact multi-valued mapping is a compact (Lindelöf) space. From this fact one can deduce at once that the composition of two continuous compact multivalued mappings is a continuous compact multi-valued mapping. Next, it is easy to prove that the Cartesian product of (compact) continuous mappings is a (compact) continuous mapping. Finally, if $T$ is closed in $X$ and $f$ is a compact continuous mapping of $X$ onto $Y$, then setting

$$g(x) = f(x) \cap T,$$

we obtain a compact continuous mapping of $X$ onto $T$.

**Theorem 1.** *The class* **A** *is* F-*hereditary, countably productive and closed under compact continuous multi-valued mappings, in particular, under continuous mappings. Every analytic space is a Lindelöf space, and consequently normal.*

Proof. The first and the third assertions follow from Definition 1 and the above remarks. The second assertion follows from the obvious fact that the topological product of a countable number of copies of $\Sigma$ is a copy of $\Sigma$. Finally, any analytic space is a Lindelöf space, for $\Sigma$ is a a Lindelöf space.

Now we shall give an internal characterization of analytic spaces.

**Definition 2.** Let $M = \{M(s)\}$ be a determining system (see 1.8) in exp $X$. An $M$-Cauchy family (in $X$) is a centered family $\mathcal{M}$ of subsets of $X$ such that $M(s) \in \mathcal{M}$ for all $s < \sigma$, where $\sigma$ is an element of $\Sigma$. The system $M$ will be called *complete* (in $X$) if the intersection of $\overline{\mathcal{M}}$ is non-void for every $M$-Cauchy family $\mathcal{M}$. An *analytic structure* in a space $X$ is a complete determining system $M$ in $X$ such that

(1) $$\mathscr{A}(M) = X.$$

In [6] the following useful result is proved:

**Lemma 1.** *A determining system $M$ in* exp $X$ *is complete in $X$ if and only if all sets*

(2) $$M(\sigma) = \bigcap_{s<\sigma} \overline{M(s)}$$

*are compact, and for any $\sigma \in \Sigma$ and open $U \supset M(\sigma)$ there exists an $s < \sigma$ with* $\overline{M(s)} \subset U$.

Now, for each $s$ in $S$, put

(3) $$\Sigma(s) = \{\sigma : s < \sigma\}.$$

Either from Lemma 1 or directly from Definition 1 it follows at once that $\{\Sigma(s)\}$ is an analytic structure in $\Sigma$.

Let $f$ be a compact, continuous multi-valued mapping of $\Sigma$ onto $X$. Set

$$M(s) = \bigcup \{f(x) : x \in \Sigma(s)\} . \tag{4}$$

By Lemma 1, $M = \{M(s)\}$ is an analytic structure in $X$. Conversely, let $M$ be an analytic structure in $X$. For each $\sigma \in \Sigma$ set

$$f(\sigma) = M(\sigma) , \tag{5}$$

where the $M(\sigma)$ are defined by (2). By Lemma 1, the multi-valued mapping $f$ is continuous. Thus we have proved the following

**Theorem 2.** *A space $X$ is analytic if and only if there exists an analytic structure in $X$.*

Now we shall prove the following

**Theorem 3.** *The following conditions on a space $X$ are equivalent:*

(a) *$X$ is analytic.*

(b) *$X$ is $\mathsf{F}(Y)$-Souslin for any $Y \supset X$.*

(c) *$X$ is $\mathbf{K}(Y)$-Souslin for some $Y \supset X$.*

Proof. First, from Lemma 1 it follows immediately that if $M$ is a complete determining system in $X$, then also $\{\overline{M(s)}\}$ is a complete determining system. Next, if $M$ is complete in $X$, then, clearly, $M$ is also complete in any $Y \subset X$. Finally, if $M$ is an analytic structure in $X$ and $Y \supset X$, then

$$\mathscr{A}(\{\overline{M(s)}^Y\}) = X .$$

Thus (a) implies (b). Obviously (b) implies (c). Finally, obviously, every determining system consisting of compact sets is complete. Thus (c) implies (a).

**Theorem 4.** *For any space we have*

$$\mathscr{A}(\mathbf{A}(X)) = \mathbf{A}(X) . \tag{6}$$

*If $X$ is analytic, then*

$$\mathbf{A}(X) = \mathscr{A}(\mathsf{F}(X)) . \tag{7}$$

Proof. The second assertion is an immediate consequence of the first and Theorems 1 and 3. By Theorem 3 and property (3), Section 1, of $\mathscr{M}$-Souslin sets, (6) is true for compact $X$. Now let $X$ be any space and let $K$ be a compactification of $X$. Since

$$\mathbf{A}(X) = X \cap \mathbf{A}(K) ,$$

(6) follows from the corresponding property of $K$.

Let $M = \{M(s)\}$ be an analytic structure in $X$. Put

$$F(s) = \bigcup \{M(\sigma) : \sigma \in \Sigma(s)\} .$$

Thus also $F = \{F(s)\}$ is an analytic structure in $X$ and

$$(8) \qquad F(s_1, \ldots, s_n) = \bigcup_{k=1}^{\infty} F(s_1, \ldots, s_n, k) .$$

Such structures will be called regular.

**Theorem 5.** *If $X$ and $Y$ are disjoint analytic subspaces of a space $T$, then there exists a Baire set $B$ in $T$ with*

$$(9) \qquad X \subset B \subset T - Y .$$

Remark. The preceding result is a generalization of the famous Luzin's first separation principle.

Corollary 1. For any space we have

$$\mathscr{B}(\mathsf{Z}(X)) \supset \text{compl. p. } \mathbf{A}(X) .$$

If $X$ is analytic, then

$$\text{compl. p. } \mathbf{A}(X) = \mathscr{B}(\mathsf{Z}(X)) .$$

Corollary 2. If $\{X_n\}$ is a disjoint sequence of analytic subspaces of $X$, then there exists a disjoint sequence $\{B_n\}$ of Baire sets of $X$ with $B_n \supset X_n$.

Proof of Theorem 5. Let $\{X(s)\}$, $\{Y(s)\}$ be regular analytic structures in $X$ and $Y$, respectively. Supposing (9) is true for no Baire set $B$, one can construct, by induction, a $\sigma$ and a $\tau$ in $\Sigma$ such that

$$(10) \qquad X(\{\sigma_1, \ldots, \sigma_n\}) \subset B \subset T - Y(\{\tau_1, \ldots, \tau_n\})$$

for no Baire set $B$ and no $n = 1, 2, \ldots$ The sets

$$X(\sigma) = \bigcap_{s < \sigma} \overline{X(s)} , \quad Y(\tau) = \bigcap_{t < \tau} \overline{Y(t)}$$

are compact and disjoint. Thus there exists a Baire set $Z$ in $T$ (in fact, a zero-set) with

$$(11) \qquad X(\sigma) \subset \operatorname{int} Z = U , \quad Y(\tau) \subset \operatorname{int} (T - Z) = V .$$

By Lemma 1 there exists an $i$ such that

$$X(\{\sigma_1, \ldots, \sigma_i\}) \subset U , \quad Y(\{\tau_1, \ldots, \tau_i\}) \subset V .$$

It follows that (10) is true for $B = Z$ and $n = i$. This contradiction establishes the Theorem 5.

**Definition 3.** A space $X$ will be called *bianalytic* if $X$ is a Baire set of some compact space.

**Theorem 6.** *The images and the preimages under perfect mappings of bianalytic spaces are bianalytic.*

Proof. Let $f$ be a perfect mapping of $X$ onto $Y$. There exists a continuous mapping $g$ of the Čech-Stone compactification $\beta(X)$ of $X$ onto $\beta(Y)$ such that $f$ is a restriction of $g$. It is well known that

$$g[\beta(X) - X] = \beta(Y) - Y .$$

By Theorem 1, $X$ (respectively, $\beta(X) - X$) is analytic if and only if $Y$ (respectively, $\beta(Y) - Y$) is such. Since clearly bianalytic space $X$ is a Baire set of $\beta(X)$ (by the Čech-Stone mapping theorem), the proof is complete.

Remark. The union of two bianalytic subspaces of a space may fail to be bianalytic. A one-to-one continuous image of a bianalytic space may fail to be bianalytic. Indeed, let $N$ be a countable infinite discrete space and let $x$ be a point of $\beta(N) - N$. Clearly, $N \cup (x)$ is a one-to-one continuous image of $N$. Next, $N$ is bianalytic and $N \cup (x)$ is not, because $\beta(N) - (N \cup (x))$ is not a Lindelöf space. Further, $N \cup (x)$ is the disjoint union of two bianalytic spaces $N$ and $(x)$.

## 3. BORELIAN SPACES

**Definition 4.** A Borelian structure in a space $X$ is a complete sequence $\{\mathcal{M}_n\}$ (see 1,13) of *countable disjoint* coverings of $X$ satisfying the following two conditions.
(1) (a) If $M_n, N_n \in \mathcal{M}_n$ and $M_k \neq N_k$ for some $k$, then

$$\bigcap_{n=1}^{\infty} \overline{M}_n \cap \bigcap_{n=1}^{\infty} \overline{N}_n = \emptyset .$$

(b) $\mathcal{M}_{n+1}$ refines $\mathcal{M}_n$, $n = 1, 2, \dots$

**Definition 5.** A space $X$ will be called *Borelian* if in $X$ there exists a Borelian structure. $\mathbf{B}$ will be used to denote the class of all Borelian spaces and $\mathbf{B}(X)$ to denote the family of all $B \subset X$, $B \in \mathbf{B}$.

From the definition we have at once the following

**Lemma 2.** *If $\{\mathcal{M}_n\}$ is a Borelian structure in $X$ and $M \in \mathcal{M}_k$, then $\{M \cap \mathcal{M}_{k+n}\}$ is a Borelian structure in $M$.*

Let $\{\mathcal{M}_n\}$ be a Borelian structure in $X$. By induction one can construct determining system $\{M(s)\}$ in $X$ such that

$$s \in S_n , \quad M(s) \neq \emptyset \Rightarrow M(s) \in \mathcal{M}_n , \tag{2}$$

$$M(s_1, \dots, s_n) = \bigcup_{k=1}^{\infty} M(s_1, \dots, s_n, k) . \tag{3}$$

Clearly $\{M(s)\}$ is an analytic structure in $X$. Let $f$ be the corresponding multi-valued mapping of $\Sigma$ onto $X$ (for the definition see Section 2, (2) and (5)). Clearly

$$\sigma \neq \tau \Rightarrow f(\sigma) \cap f(\tau) = \emptyset . \tag{4}$$

Any multi-valued mapping satisfying (4) will be called *disjoint*.

Conversely, let $f$ be a compact, continuous and disjoint multi-valued mapping of $\Sigma$ onto $X$. Put

$$M(s) = \{f(\sigma) : s \prec \sigma\} .$$

It is easy to see that $\{\mathcal{M}_n\}$,

(5) $$\mathcal{M}_n = \{M(s) : s \in S_n\},$$

is a Borelian structure in $X$.

By Lemma 1, $\{M(s)\}$ is an analytical structure in $X$. Clearly, all $\mathcal{M}_n$ are disjoint. Thus $\{\mathcal{M}_n\}$ is a complete sequence of disjoint coverings. Finally, let $\sigma \neq \tau$. The sets $f(\sigma)$ and $f(\tau)$ being compact and disjoint, we can choose disjoint open sets $U$ and $V$ with

$$f(\sigma) \subset U, \quad f(\tau) \subset V.$$

By Lemma 1, there exist $s, t \in S_n$, $s \prec \sigma$, $t \prec \tau$, with $\overline{M(s)} \subset U$ and $\overline{M(t)} \subset V$. Thus (a) is fulfilled. Clearly also (b) is fulfilled. We have proved the following

**Theorem 7.** *A space $X$ is Borelian if and only if there exists a compact and continuous disjoint multi-valued mapping of $\Sigma$ onto $X$.*

**Theorem 8.** *The class of all Borelian spaces is F-hereditary, countably productive and closed under compact and continuous disjoint multi-valued mappings. In particular, one-to-one continuous images and perfect preimages of Borelian spaces are Borelian.*

The proof follows at once from Theorem 7 and from the properties of multi-valued mappings.

**Lemma 3.** *Let $f$ be a compact and continuous disjoint multivalued mapping of $X$ onto $Y$. There exists a space $T$, a one-to-one continuous mapping $g$ of $T$ onto $Y$, and a perfect mapping $h$ of $T$ onto a closed subset of $X$, such that*

(6) $$f = g \circ h^{-1}$$

*where $h^{-1}$ is the inverse of $h$,* i. e.

$$h^{-1}(x) = h^{-1}[(x)].$$

*Conversely, if $g$ is one-to-one continuous and $h$ is perfect, then the mapping $f$ (given by* (6)) *is a compact and continuous disjoint multi-valued mapping.*

Proof. The second assertion is obvious. Let $f$ be a mapping which satisfies the assumptions of the first assertion. Let $\mathfrak{T}$ be the smallest topology in the set $Y$ containing the topology of the space $Y$ (that means, open sets of $Y$) and all sets $f[U]$ with $U$ open in $X$, where, of course,

(7) $$f[U] = \bigcup \{f(x) : x \in U\}.$$

Let $T$ be the set $Y$ with the topology $\mathfrak{T}$. Clearly the identity mapping $g$ of $T$ onto $Y$ is continuous. Now, since $f$ is disjoint, for any $t$ in $T$ there exists one and only one point $x$ of $X$ with $t \in f(x)$. Put $h(t) = x$. Clearly (5) holds. It remains to prove that $h$ is perfect. First, $h$ is continuous, because, by definition of $\mathfrak{T}$, all sets of the form

$$g^{-1}[U] = f[U],$$

where $U$ is open in $X$, belong to $\mathfrak{T}$. Since $f$ is compact and $g^{-1}(x) = f(x)$, the inverses of points are compact. Finally, if $F$ is closed in $T$, then $T - F = U$ is open in $T$. To prove $h[F]$ is closed in $X$ it is sufficient to show that

$$V = f^{-1}[U] = \{x : f(x) \subset U\}$$

is open in $X$. Let $x \in V$. By definition of the topology $\mathfrak{T}$ of $T$, there exists an open set $V_1$ in $X$ and open set $U_1$ in $Y$ with

$$f(x) \subset U_1 \cap f[V_1] \subset U .$$

By continuity of $f$ the set $f^{-1}[U_1]$ is open in $X$ and by definition of the topology of $T$, also $V_1 = f[f^{-1}[V]]$ is open in $X$. Clearly

$$x \in V_1 \cap f^{-1}[U_1] \subset f^{-1}[U] .$$

Thus $x$ is an interior point of $f^{-1}[U]$. Since $x$ is arbitrary, $f^{-1}[U]$ is open. The proof is complete.

In view of Lemma 3, Theorem 7 may be restated as follows.

**Theorem 9.** *A space $X$ is Borelian if and only if $X$ is a one-to-one continuous image of a space which admits a perfect mapping onto a closed subspace of the space $\Sigma$ of irrational numbers.*

Now we shall investigate $\mathbf{B}(X)$ for any $X$. First we shall prove the following

**Theorem 10.** *For any space $X$ we have*

$$\mathscr{B}_*(\mathbf{B}(X)) = \mathbf{B}(X) . \tag{8}$$

Proof. We must show that $\mathbf{B}(X)$ is closed under countable intersections and countable disjoint unions. Let $\{Y_k\}$ be a disjoint sequence of Borelian subspaces of $X$ and let $\{\mathscr{M}_n(k)\}_{n=1}^{\infty}$ be a Borelian structure in $Y_k$. It is easy to see that $\{\mathscr{M}_n\}$ is a Borelian structure in the union of all $X_k$, where

$$\mathscr{M}_n = \bigcup_{k=1}^{\infty} \mathscr{M}_n(k) .$$

The only one, perhaps, not entirely trivial point is the validity of condition (a) in Definition 4. We must show that

$$\bigcap_{k=1}^{\infty} \overline{M}_n^X = \bigcap_{n=1}^{\infty} \overline{M}_n^{Y_k} ,$$

where $M_n \in \mathscr{M}_n(k)$, $n = 1, 2, \ldots$ But this follows from Lemma 1, for any Borelian structure may be considered as an analytical structure (see the first part of the proof of Theorem 7).

Now let $\{Y_k\}$ be an arbitrary sequence of Borelian subspaces of $X$. Let $Y$ be the intersection of all $Y_k$ and let $\{\mathscr{M}_n(k)\}_{n=1}^{\infty}$ be a Borelian structure in $Y_k$. Clearly

$$\{Y \cap \mathscr{M}_n(k)\} \tag{9}$$

is a complete collection of countable coverings of $Y$. Let $\mathscr{M}_i$ be the family of all sets of the form

$$Y \cap \bigcap \{M_n(k) : k \leqq i, n \leqq i\}$$

where $M_n(k) \in \mathcal{M}_n(k)$. Obviously, $\{\mathcal{M}_i\}$ is a complete sequence of countable disjoint coverings of $Y$, which satisfies condition (b) of Definition 4. To prove that condition (a) is also satisfied, let us suppose $M_n$, $N_n \in \mathcal{M}_n$, $n = 1, 2, \ldots$, and $M_k \neq N_k$. By construction of the sequence $\{\mathcal{M}_n\}$, there exist decreasing sequences $\{N_n(i)\}_{n=1}^{\infty}$, $\{\mathcal{M}_n(i)\}_{n=1}^{\infty}$, $i = 1, 2, \ldots$, where $N_n(i)$, $M_n(i) \in \mathcal{M}_n(i)$, such that

$$(10) \qquad M_n(n) \supset M_n\,, \quad N_n(n) \supset N_n\,.$$

If all $N_n$ and $M_n$ are non-void, the sequences $\{M_n(i)\}$, $\{N_n(i)\}$ are uniquely determined by (10) and

$$(11) \qquad \begin{aligned} M_n &= \bigcap \{M_i(j) : i \leqq n,\ j \leqq n\}\,, \\ N_n &= \bigcap \{N_i(j) : i \leqq n,\ j \leqq n\}\,. \end{aligned}$$

Since $M_k \neq N_k$, there exist an $i$ and an $j \leqq k$, with

$$M_i(j) \neq N_i(j)\,.$$

$\{\mathcal{M}_j\}$ being a Borelian structure, we have

$$\bigcap_{n=1}^{\infty} \overline{M}_n(j) \cap \bigcap_{n=1}^{\infty} \overline{M}_n(j) = \emptyset\,,$$

and consequently, in view of (11), (1) holds. The proof is complete.

Remark. In general the family $\mathbf{B}(X)$ is not closed under countable unions. Even a $\sigma$-compact space may fail to be a Borelian space. For example, let $K$ be an uncountable compact space with only one accumulation point, say $x$, and let $X$ be the topological product of $K$ and the discrete space $N$ of positive integers. Identifying the points of the set $(x) \times N = y$ we obtain the quotient space $Y$. We shall prove that $Y$ is not a Borelian space. Let us suppose that $\{\mathcal{M}_n\}$ is a Borelian structure $Y$. By Lemma 2, any set from

$$(12) \qquad \mathcal{M} = \bigcup_{n=1}^{\infty} \mathcal{M}_n$$

is a Borelian space, and consequently, a Lindelöf space. Thus any $M \in \mathcal{M}$ either contains $y$ or is countable. Therefore the set

$$Y_0 = \bigcup \{M : M \in \mathcal{M},\ y \notin M\}$$

must be countable. The set $Y - Y_0$ is, by definition of Borelian structures, compact. But this is impossible for we can choose

$$y_n \in (K \times (n)) - Y_0$$

and clearly, the set of all $y_n$ has no accumulation point in $Y$.

**Theorem 11.** *For any space $X$ we have*

$$(13) \qquad \mathbf{B}(X) \subset \mathcal{B}_*(\mathsf{F}(X) \cap \mathcal{B}(\mathsf{Z}(X)))\,.$$

*More precisely, $\mathbf{B}(X)$ is contained in the family consisting of all countable intersections of countable disjoint unions of sets of the form $F \cap Z$, $F \in \mathsf{F}(X)$, $Z \in \mathcal{B}(\mathsf{Z}(X))$ and obviously this family is contained in the right side of* (13).

Proof. Let $Y$ be a Borelian subspace of $X$ and let $\{\mathcal{M}_n\}$ be a Borelian structure in $Y$. By Lemma 2, all sets from (12) are Borelian spaces, and consequently, by Corollary 2 to Theorem 5, there exist Baire sets $B(M) \supset M$ in $X$ such that the families

$$\{B(M) : M \in \mathcal{M}_n\} \tag{14}$$

are disjoint. Put

$$F(M) = B(M) \cap \overline{M}^X .$$

Since $\{\mathcal{M}_n\}$ is a complete sequence, we have

$$\bigcup_{\{M_n\}} \bigcap_{n=1}^{\infty} \overline{M}_n = Y , \tag{15}$$

where $\{M_n\}$ runs over all sequences with $M_n \in \mathcal{M}_n$. The coverings $\{F(M) : M \in \mathcal{M}_n\}$ are disjoint, because the coverings (14) are disjoint, and consequently,

$$\bigcup_{\{M_n\}} \bigcap_{n=1}^{\infty} F(M_n) = \bigcap_{n=1}^{\infty} \bigcup \{F(M); M \in \mathcal{M}_n\} . \tag{16}$$

Clearly the set on the right side of (16) contains $X$ and the set on the left side of (16) is contained in the left side of (15), and consequently, in $X$. It follows that

$$\bigcap_{n=1}^{\infty} \bigcup \{F(M) : M \in \mathcal{M}_n\} = Y ,$$

which establishes Theorem 11.

Remark. The above proof of Theorem 11 depends essentially upon the theory of analytic spaces. Indeed, the existence of (14) follows from Theorem 5. Making use of the same trick as in the proof of Theorem 5 one can prove the existence of Baire sets (14) directly.

Now we shall prove the following useful

**Theorem 12.** *If $X$ is a Borelian space, then*

$$\mathbf{B}(X) = \mathcal{B}_*(\mathsf{F}(X) \cup \mathcal{B}(\mathsf{Z}(X))) . \tag{17}$$

*More precisely,* $\mathbf{B}(X)$ *coincides with the family described in Theorem* 11.

Proof. By Theorem 11 the inclusion $\subset$ holds. To prove the converse inclusion, by Theorem 10 it is sufficient to prove

$$X \in \mathbf{B} \Rightarrow \mathcal{B}(\mathsf{Z}(X)) \subset \mathbf{B}(X) . \tag{18}$$

It is easy to see that (18) follows from

$$Y \text{ compact} \Rightarrow \mathcal{B}(\mathsf{Z}(Y)) \subset \mathbf{B}(Y) . \tag{19}$$

Indeed, let $Y$ be the Čech-Stone compactification of a Borelian space $X$. Since any bounded real-valued continuous functions on $X$ has a continuous extension on $Y$, we have

$$\mathsf{Z}(X) = X \cap \mathsf{Z}(Y)$$

and consequently,

$$\mathcal{B}(\mathsf{Z}(X)) = X \cap \mathcal{B}(\mathsf{Z}(Y)) . \tag{20}$$

On the other hand obviously

$$\mathbf{B}(X) = \{B : B \subset X,\ B \in \mathbf{B}(Y)\}\,. \tag{21}$$

Combining (19), (20) and (21) we obtain

$$\mathscr{B}(\mathsf{Z}(X)) \subset \mathbf{B}(X) \tag{22}$$

which establishes (18). It remains to prove (19). This follows from lemmas 4 and 5.

**Lemma 4.** *Every cozero-set $N$ of a compact space $Y$ is a Borelian space.*

Proof. Let $f$ be a real-valued continuous function on $Y$ with

$$N = \{x : f(x) \neq 0\}\,.$$

Let $g$ be the restriction of $f$ to $N$. The mapping $g$ is perfect because $f$ is such. By Theorem 8 it is sufficient to show that $g[N]$ is a Borelian space. Let $E$, $R$, $I$ denote the spaces of all real, rational and irrational numbers, respectively. The subspace $R \cap \cap\, g[N]$ is countable, and hence, Borelian. The set $g[N] \cap I$ is closed in $I$ because

$$\overline{g[N]}^E \subset I \cup (0)$$

and $0 \notin I$. By Theorem 8, the space $g[N] \cap I$ is Borelian. By Theorem 10, the space

$$g[N] = (g[N] \cap R) \cup (g[N] \cap I)$$

is Borelian.

**Lemma 5.** *Let* $\mathsf{N}(X)$ *denote the family of all cozero-sets of a space $X$. Then*

$$\mathscr{B}_*(\mathsf{N}(X)) = \mathscr{B}(\mathsf{N}(X)) = \mathscr{B}(\mathsf{Z}(X))\,. \tag{23}$$

Proof. I cannot prove (23) without the Borel classification of Baire sets. A proof, making use of the Borel classification, may be found, for example, in [13], p. 255.

Corollary 1. Every bianalytic space is Borelian.

Corollary 2. If $X$ is a perfectly normal Borelian space, in particular a metrizable Borelian space, then

$$\mathbf{B}(X) = \mathscr{B}(\mathsf{Z}(X))\,. \tag{24}$$

Remark. Conversely, from (24) it follows at once that $X$ is perfectly normal. Combining Corollary 2 of Theorem 12 and Corollary 1 of Theorem 5 we obtain

$$\mathbf{B}(X) = \mathscr{B}(\mathsf{Z}(X)) = \text{compl. p. } \mathbf{A}(X) \tag{25}$$

for any perfectly normal space $X$.

**Theorem 13.** *The class of all Borelian spaces is the smallest class of spaces containing all bianalytic spaces and closed under one-to-one continuous mappings.*

Proof. By Corollary 1 every bianalytic space is Borelian and by Theorem 8 a one-to-one continuous image of a Borelian space is a Borelian space. Thus the class of all one-to-one continuous images of bianalytic spaces is contained in $\mathbf{B}$. By Theorem 9, if $X$ is a Borelian space, then there exists a space $T$, a one-to-one continuous mapping of $T$ onto $X$, and a perfect mapping of $T$ onto a closed subspace $B$ of $\Sigma$. The space $B$ is bianalytic because $B$ is a $G_\delta$ in the closed unit interval of real numbers, and open sets of metrizable spaces are Baire sets. By Theorem 6, the space $T$ is bianalytic.

**Theorem 14.** *$Y$ is a Borelian space if and only if there exists a complete sequence $\{\mathcal{M}_n\}$ of countable disjoint coverings of $Y$ consisting of analytic subspaces of $Y$.*

Proof. By Lemma 2, if $\{\mathcal{M}_n\}$ is a Borelian structure in $Y$, then all sets belonging to the union of all $\mathcal{M}_n$ are Borelian spaces, and consequently, analytic spaces. Conversely, let $\{\mathcal{M}_n\}$ be a complete sequence of countable disjoint coverings of $Y$ consisting of analytic subspaces of $Y$. By the proof of Theorem 11, for any space $X$ containing $Y$ we have

$$Y \in \mathcal{B}_*(\mathsf{F}(X) \cup \mathcal{B}(\mathsf{Z}(X)))\,. \tag{26}$$

By Theorem 12, $Y$ is a Borelian space.

In [9] other proofs of the theorems of this section are sketched. The space $\Sigma$ and multi-valued mappings may be eliminated. Using Borelian structures one can prove Theorem 9 directly. Theorem 9 follows from the following two lemmas.

**Lemma 6.** *$X$ is the preimage under a perfect mapping of a closed subspace of $\Sigma$ if and only if there exists a Borelian structure $\{\mathcal{M}_n\}$ in $X$ such that all sets belonging to*

$$\mathcal{M} = \bigcup_{n=1}^{\infty} \mathcal{M}_n \tag{27}$$

*are open.*

**Lemma 7.** *Any Borelian space is an one-to-one continuous image of a space which has a Borelian structure $\{\mathcal{M}_n\}$ such that the sets belonging to (27) are open.*

Next, Theorem 10 and 11 depend neither on $\Sigma$ nor on multi-valued mappings. Moreover, one can prove Theorem 11 without the theory of analytic spaces (see the Remark following the proof of Theorem 11).

In the proof of Theorem 12 the space $\Sigma$ is used to prove that any cozero-set of a compact space is a Borelian space. We need only the fact that $\Sigma$ is a Borelian space. In this case the use of $\Sigma$ is very convenient.

Making use of the following lemma and Lemma 7 one can prove Theorem 12 without Theorem 9.

**Lemma 8.** *Let $\{\mathcal{M}_n\}$ be a complete sequence of disjoint open coverings of a space $X$. Then*

$$X = \bigcap_{n=1}^{\infty} \bigcup \overline{\mathcal{M}}_n^K \tag{28}$$

*where $K$ is the Čech-Stone compactification of $X$, and all sets from $\bigcup_{n=1}^{\infty} \overline{\mathcal{M}}_n^K$ are closed and open in $K$, in particular, all these sets are zero-sets in $K$.*

Proof. The sets from (27) are closed in $X$. Thus their closures in $K$ are open and closed. Clearly all the families $\overline{\mathcal{M}}_n^K$ are disjoint. It follows that

$$\bigcap_{n=1}^{\infty} \bigcup \overline{\mathcal{M}}_n^K = \bigcup_{\{M_n\}} \bigcap_{n=1}^{\infty} \overline{M}_n^K\,. \tag{29}$$

But the set on the right side of (29) is $X$ and hence (28) holds.

We conclude this section by the following

**Theorem 15.** *A space $X$ is analytic if and only if $X$ is a continuous image of a bianalytic space. Moreover, every analytic space is a continuous image of a space which admits a perfect mapping onto a closed subspace of $\Sigma$.*

The complete proof can be found in [6]. Here we give only a suggestion. Let $\{M(s)\}$ be an analytic structure in $X$ and let $K$ be a compactification of $X$. Let $\pi$ be the projection of the product space $Y = K \times \Sigma$ onto $K$. Consider the subspace

$$T = \bigcap_{n=1}^{\infty} \bigcup \{\overline{M(s)} \times \Sigma(s) : s \in S_n\}$$

of $Y$. It is easy to see that $\pi[T] = X$ and that $\{\mathcal{M}_n\}$ is a Borelian structure in $T$, such that all sets belonging to (27) are open, where

$$\mathcal{M}_n = T \cap \{\overline{M(s)} \times \Sigma(s) : s \in S_n\} .$$

## 4. METRIZABLE BORELIAN AND ANALYTIC SPACES

For convenience, metrizable Borelian (analytic) spaces will be called *classical Borelian* (*classical analytic*) spaces. Since every metrizable Lindelöf space is separable, every classical analytic space is separable.

**Theorem 16.** *The following conditions on a metrizable space $X$ are equivalent:*

(a) *$X$ is Borelian.*

(b) *$X$ is bianalytic.*

(c) *$X$ is a Baire set of some complete metrizable separable space.*

(d) *$X$ is separable, and if $X$ is contained in a metrizable space $Y$, then $X$ is a Baire set of $Y$.*

In the classical theory the following theorem plays the fundamental rôle:

**Theorem 17.** *Every classical Borelian space is an one-to-one continuous image of a closed subspace of the space $\Sigma$ of irrational numbers. Every classical analytic space is a continuous image of $\Sigma$.*

The classical theory makes use of this theorem instead of Borelian and analytic structures. Thus this theorem loses its importance in our presentation.

First we shall prove the following two results:

**Theorem 18.** *A space $X$ is a continuous image of $\Sigma$ if and only if there exists an analytical structure $\{M(s)\}$ in $X$ such that the sets*

(1) $M(\sigma) = \bigcap_{s < \sigma} \overline{M(s)}$

*contain at most one point.*

**Theorem 19.** *A space $X$ is an one-to-one continuous image of a closed subspace of $\Sigma$ if and only if there exists a Borelian structure $\{\mathcal{M}_n\}$ in $X$ such that*

(2) *the sets $\bigcap_{n=1}^{\infty} \overline{M}_n$ contain at most one point.*

Proof. In both theorems the conditions are clearly necessary. Conversely, let $f$ be the multivalued mapping of $\Sigma$ onto $X$ corresponding to the analytic structure $\{M(s)\}$. Let $F$ be the set of all $\sigma$ with non-void images. Since $f$ is continuous, $F$ is closed in $\Sigma$. For each $\sigma$ in $F$ let $g(\sigma)$ be the element of $f(\sigma)$. Then $g$ is a continuous mapping of $F$ onto $X$. It is well known and easy to prove that any closed subset of $\Sigma$ is a continuous image of $\Sigma$, moreover, a retract of $\Sigma$. The proof of Theorem 18 is complete. The proof of sufficiency of the condition of Theorem 19 can be proved analoguously.

Remark. The fact that every non-void closed subset $F$ of $\Sigma$ is a retract of $\Sigma$ can be proved as follows:

There exists a mapping $f$ of $S$ to $F$ such that $f(s) \in F \cap \Sigma(s)$ if possible. Put $g(\sigma) = \sigma$ for $\sigma \in F$. If $\sigma \notin F$ and $\Sigma(\sigma_1) \cap F = \emptyset$, put $g(\sigma) = f(\sigma_1)$. In the remaining case there exists an $n$ with

$$\Sigma(\{\sigma_1, \ldots, \sigma_n\}) \cap F \neq \emptyset\,, \quad M(\{\sigma_1, \ldots, \sigma_{n+1}\}) \cap F = \emptyset\,.$$

Put $g(\sigma) = f(\{\sigma_1, \ldots, \sigma_n\})$. It is easy to see that $g$ is a retraction of $\Sigma$ to $F$. Indeed, if $F \cap \Sigma(s) \neq \emptyset$, then $g[\Sigma(s)] \subset F \cap \Sigma(s)$ and in the other case $g[M(s)]$ is a one point set of $F$.

The part of Theorem 17 concerning classical analytic spaces follows easily from 18. Indeed, if $\{M(s)\}$ is an analytical structure in a metrizable space $X$ and if $\varrho$ is a metric generating the topology of $X$, then one can construct a determining system $\{F(s)\}$ in $X$, such that $\mathscr{A}(F) = X$ and

(a) if $s \in S_n$, then the diameter of $F(s)$ is less than $1/n$.

(b) $\{F(s)\}$ is a refinement of $\{M(s)\}$, i. e. for each $\sigma \in \Sigma$ there exists a $\tau$ in $\Sigma$ such that

$$F(\{\tau_1, \ldots, \tau_n\}) \subset M(\{\sigma_1, \ldots, \sigma_n\})\,.$$

By (b), $\{F(s)\}$ is an analytic structure in $X$ and by (a) the sets $F(\sigma)$ contain at most one point.

The proof of the second part of Theorem 17 is more difficult. Let $\mathbf{B}_1$ be the class of all one-to-one continuous images of closed subspaces of $\Sigma$ and let $\mathbf{B}_1(X) = \{Y : Y \subset X,\ Y \in \mathbf{B}_1\}$. Using Borelian structures satisfying (2), we obtain at once

(3) $\mathscr{B}_*(\mathbf{B}_1(X)) = \mathbf{B}_1(X)\,.$

Indeed, the proof of Theorem (10) is applicable. Next

(4) $\mathbf{B}_1$ is countably productive and F-hereditary. Indeed, the topological product of a countable number of copies of $\Sigma$ is homeomorphic to $\Sigma$. From (3) and (4) one can deduce at once

(5) Any complete metrizable separable space belongs to $\mathbf{B}_1$.

Indeed, clearly the Euclidean line belongs to $\mathbf{B}_1$ (as a union of $\Sigma$ and a coutable set); by (4) closed subspace of the topological product of the coutable number of Euclidean lines belong to $\mathbf{B}_1$. Finally, it is well known that any complete metrizable separable space is homeomorphic with a closed subspace of this topological product.

Now let $K$ be a metrizable compact space. Every open subspace of $K$ is a complete metrizable separable space, and hence, every open subspace of $K$ belongs to $\mathbf{B}_1(K)$. By Lemma 5 (Section 3) and (3) we have

$$\mathbf{B}_1(K) \supset \mathscr{B}(\mathsf{Z}(K)) .$$

We have proved that Baire sets of compact metrizable spaces belong to $\mathbf{B}_1$. The proof is complete.

## 5. REMARKS

Classical Borelian spaces are precisely those spaces which belong to

$$\mathscr{B}(\mathbf{K}(X)) \tag{1}$$

for some metrizable space. It seems that this fact led V. Šnejder to introduce the spaces which belong to (1) for some space $X$ (called K-Borelian by G. Choquet) and their continuous images which coincide by analytic spaces (by Theorem 15).

In [2] G. Choquet showed the relation between analytic spaces ($K$-analytic in his terminology) and the so-called $K$-Souslin spaces (spaces which are $\mathbf{K}(X)$-Souslin for some $X$). In [3] he proved the essential part of Theorem 15 and showed that a metrizable space $X$ is analytic if and only if $X$ is a classical analytic space (using classical results). M. Sion independently reproved all Choquet's results from [3]. Further, he tried to prove the invariance of K-Borelian spaces under one-to-one continuous mappings. He proved that a one-to-one continuous image of a K-Borelian space $X$ is K-Borelian under certain drastic assumption on a $\sigma$-compact space containing $X$. It seems that this problem is still unsolved.

In [8] the author introduced analytic structures, and using these, reproved all Choquet's results and proved some, it seems, new results. In [10] he introduced bianalytic spaces, proved Theorem 5 and gave the first internal characterization of classical Borelian spaces. In [9] and [6] somewhat other proofs of some results of the present note are given.

### References


[1] *Bourbaki*: Topologie générale. Chapitre 9. Paris 1958.
[2] *G. Choquet*: Theory of Capacities. Ann. Inst. Fourier, *5* (1953–1954), 131–295.
[3] *G. Choquet*: Ensembles $K$-analytiques et $K$-Sousliniens. Ann. Inst. Fourier, *IX* (1959), 75–81.
[4] *E. Čech*: On Bicompact Spaces. Ann. of Math. *38* (1937), 823–844.
[5] *Z. Frolík*: On Almost Realcompact Spaces. Bull. Acad. Pol., *9* (1961), 247—250.
[6] *Z. Frolík*: On Analytic Spaces. Bull. Acad. Pol., *9* (1961), 721—725.
[7] *Z. Frolík*: A Generalization of Realcompact Spaces. To appear in Czech. Math. J.
[8] *Z. Frolík*: On Descriptive Theory of Sets and Spaces. To appear in Czech. Math. J.
[9] *Z. Frolík*: On Borelian and Bianalytic Spaces. Czech. Math. J. *11(86)*, (1961), 629—631.
[10] *Z. Frolík*: On Bianalytic Spaces. To appear in Czech. Math. J.
[11] *Z. Frolík*: Generalizations of the $G_\delta$-property of Complete Metric Spaces. Czech. Math. J. *10* (85), 1960, 359–379.

[12] *M. Katětov*: Measures in Fully Normal Spaces. Fund. Math. *38* (1951), 73–84.
[13] *K. Kuratowski*: Topologie I. Warszawa 1952.
[14] *M. Sion*: On Analytic Sets in Topological Spaces. Trans. Amer. Math. Soc., *96* (1960), 341–354.
[15] *M. Sion*: Topological and Measure Theoretic Properties of analytic sets. Proc. Amer. Math. Soc. *11* (1960), 769–776.
[16] *V. E. Šnejder*: Continuous Images of Souslin and Borel Sets. Dokl. Akad. Nauk SSSR *50* (1945), 77–79.
[17] *V. E. Šnejder*: Descriptive Theory of Sets in Topological Spaces. Ibid. 81–83.