A. Goetz

A notion of uniformity for L-spaces of Frechet

In: (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the symposium held in Prague in September 1961. Academia Publishing House of the Czechoslovak Academy of Sciences, Prague, 1962. pp. [177]--178.

Persistent URL: http://dml.cz/dmlcz/700926

A NOTION OF UNIFORMITY FOR L-SPACES OF FRECHET

A. GOETZ

Wrocław

Let be given a set X and a relation $\xi\, \boldsymbol{n}\, \xi'$ between sequences $\xi = \{x_n\}$ and $\xi' = \{x'_n\}$ of elements of X (called nearness relation) which satisfies the following conditions:

(i) $\xi\, \boldsymbol{n}\, \xi$;

(ii) if $\xi\, \boldsymbol{n}\, \xi'$, then $\xi'\, \boldsymbol{n}\, \xi$;

(iii) if $\xi\, \boldsymbol{n}\, \xi'$ and $\xi'\, \boldsymbol{n}\, \xi''$, then $\xi\, \boldsymbol{n}\, \xi''$;

(iv) $\{x\}\, \boldsymbol{n}\, \{x'\}$ if and only if $x = x'$;[1])

(v) if $\{x_i\}\, \boldsymbol{n}\, \{x'_i\}$, then $\{x_{i_n}\}\, \boldsymbol{n}\, \{x'_{i_n}\}$ for every sequence $\{i_n\}$ of indices;

(vi) if every sequence $\{i_k\}$ of natural numbers contains a subsequence $\{i'_n\}$, such that $\{x_{i'_n}\}\, \boldsymbol{n}\, \{x'_{i'_n}\}$, then $\{x_n\}\, \boldsymbol{n}\, \{x'_n\}$.

X with the relation $\boldsymbol{n}$ is called a UL^*-space. The relation $\boldsymbol{n}$ is called sometimes a UL^*-structure in X.

By setting $x = \lim x_n$ iff $\{x_n\}\, \boldsymbol{n}\, \{x\}$, X becomes an L^*-space of Fréchet.

A natural order may be introduced into the set of all UL^*-structures of the given set X by setting $\boldsymbol{n} \leqq \boldsymbol{m}$ iff $\xi\, \boldsymbol{n}\, \xi'$ implies $\xi\, \boldsymbol{m}\, \xi'$.

Theorem. *The set of all UL^*-structures of X form an absolutely multiplicative semilattice. Its subsemilattice consisting of all UL^*-structures, which induce the same convergence, is a lattice and contains the least and the largest elements.*

Theorem. *The lattice of UL^*-structures of X inducing a convergence, for which X is compact, contains a single element.*

The UL^*-structure enables to introduce for Fréchet spaces some notions known for metric spaces or generally for uniform spaces: the notion of uniform convergencs of sequences of functions, of uniform continuity of functions, of Cauchy sequencee and completeness.

A function $f(x)$ is called uniformly continuous if $\{x_n\}\, \boldsymbol{n}\, \{x'_n\}$ implies $\{f(x_n)\}\, \boldsymbol{m}\, \{f(x'_n)\}$, where $\boldsymbol{n}$ is the nearness relation in the domain of the function and $\boldsymbol{m}$ the nearness relation in the set of values of the function.

A sequence $\{f_n\}$ of functions with values in an UL^*-space is said to converge uniformly to f if for each sequence $\{x_n\}$ $\{f_n(x_n)\}\, \boldsymbol{m}\, \{f(x_n)\}$.

---

[1]) $\{x\}$ denotes the constant sequence ($x_n = x$, $n = 1, 2, \ldots$).

For functions defined in a UL^*-space X with values in a metric space (or more generally in a special kind of UL^*-spaces) the following theorem holds.

Theorem. *The limit of a uniformly convergent sequence of uniformly continuous functions is a uniformly continuous function.*

A sequence $\{x_n\}$ is called a Cauchy sequence if $\{x_n\} \boldsymbol{n} \{x_{i_n}\}$ for every subsequence $\{x_{i_n}\}$. X is complete iff all Cauchy sequences are convergent.

The questions of completeness and completion of UL^*-spaces are still open.

The paper is to be published in "Colloquium Mathematicum", *9* (1962).