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First some definitions and notations, introduced elsewhere (cf. [1]), will be recalled:

(i) X is a completely regular topological space,

(ii) $\mathfrak{a}$ is an infinite cardinal number,

(iii) $\mathscr{B}_{\mathfrak{a}}(X)$ is the class of all locally finite open coverings of X of the form $(U_\iota)_{\iota \in I}$, each of which fulfils the following conditions:

$$\text{card}\,(I) \leqq \mathfrak{a}$$

(i.e., the cardinal number of the index set I does not exceed $\mathfrak{a}$); there exists a closed covering $(F_\iota)_{\iota \in I}$ of X such that, for each $\iota \in I$, the two closed subsets F_ι and $X - U_\iota$ are completely separated (according to [5]),

(iv) $\mathscr{A}_{\mathfrak{a}}(X)$ is the uniform structure on X having as a fundamental system of "entourages" (i. e., a base of the filter of "entourages") in the BOURBAKI sense, the set of all subsets B of $X \times X$ of the form

$$B = \bigcup_{\iota \in I} (U_\iota \times U_\iota)\,.$$

Because of lemma 2, § 8 and def. 4, § 3 of [1], the above uniform structure $\mathscr{A}_{\mathfrak{a}}(X)$ coincides with the "$\mathfrak{a}$-struttura uniforme" of [1] def. 4, § 3. Therefore, in view of prop. 3, § 3 of [1], the uniform structure $\mathscr{A}_{\mathfrak{a}}(X)$ agrees with the underlying topology of the completely regular space X.

Now, consider the completion $\upsilon_{\mathfrak{a}}(X)$ of X under the uniform structure $\mathscr{A}_{\mathfrak{a}}(X)$. It is well-known that

The completely regular space $\upsilon_{\mathfrak{a}}(X)$ together with a certain standard uniform structure $\mathscr{U}$ is complete, and the couple $(\upsilon_{\mathfrak{a}}(X), \mathscr{U})$ is related to the couple $(X, \mathscr{A}_{\mathfrak{a}}(X))$ in the following way: there exists an injection j of X in $\upsilon_{\mathfrak{a}}(X)$ such that

1. *$j(X)$ is dense in $\upsilon_{\mathfrak{a}}(X)$,*

2. *j is a uniform isomorphism of X equipped with $\mathscr{A}_{\mathfrak{a}}(X)$ onto $j(X)$ equipped with the uniform structure $\mathscr{U}_{j(X)}$ induced by $\mathscr{U}$ on $j(X)$.*

Next we construct the class $\mathscr{B}_{\mathfrak{a}}(\upsilon_{\mathfrak{a}}(X))$ of coverings of $\upsilon_{\mathfrak{a}}(X)$ (as previously defined), and then construct the uniform structure $\mathscr{A}_{\mathfrak{a}}(\upsilon_{\mathfrak{a}}(X))$ in the same manner as $\mathscr{A}_{\mathfrak{a}}(X)$.

Theorem 1. *The following equality holds:*

$$\mathscr{U} = \mathscr{A}_{\mathfrak{a}}(\upsilon(X_{\mathfrak{a}}))\,.$$

This result stated, the following definition is consistent:

Definition. A completely regular space X is said to be $\mathfrak{a}$-*complete* if and only if X, equipped with $\mathscr{A}_{\mathfrak{a}}(X)$, is a complete uniform space.

Theorem 1 provides many $\mathfrak{a}$-complete spaces; e. g. all $\upsilon_{\mathfrak{a}}(X)$. We shall return later to this point. For the moment we observe a few facts concerning $\mathfrak{a}$-completeness.

Note that, in [2], $\mathfrak{a}$-complete spaces were called "spazii $\mathscr{A}_{\mathfrak{a}}$-completi".

Lemma 1. *If M is a metrizable space having a base for its topology whose cardinal number does not exceed $\mathfrak{a}$ then M is $\mathfrak{a}$-complete (in the sense stated).*

Indeed any paracompact space having a base whose cardinal number does not exceed $\mathfrak{a}$ is $\mathfrak{a}$-complete in the sense stated.

Proposition 1. *If F is a closed subset of an $\mathfrak{a}$-complete space X, then the subspace F is also $\mathfrak{a}$-complete.*

Proposition 2. *If $(X_\lambda)_{\lambda \in L}$ is an unrestricted family of $\mathfrak{a}$-complete spaces then its product $\prod_{\lambda \in L} X_\lambda$ is also $\mathfrak{a}$-complete.*

Proposition 3. *If $(A_\lambda)_{\lambda \in L}$ is a family of subsets of a completely regular space X and, for each $\lambda \in L$, the subspace A_λ is $\mathfrak{a}$-complete then the intersection $\bigcap_{\lambda \in L} A_\lambda$ is also $\mathfrak{a}$-complete.*

Next we shall state a proposition which, as will be seen later, relates $\mathfrak{a}$-completeness to a known property of topological spaces.

Proposition 4. *A necessary and sufficient condition for a space X to be an $\mathfrak{a}$-complete space is that X be a closed subset of a product of metric spaces each having a base for its topology whose cardinal number does not exceed $\mathfrak{a}$.*

Of course, if we take

$$\omega = \text{card}(\mathbf{N}),$$

where $\mathbf{N}$ is the set of non negative integers, then it turns out that $\mathfrak{a}$-complete spaces are those and only those which are Q-spaces in the sense of HEWITT (realcompact spaces, according to [5]). Therefore $\mathfrak{a}$-completeness may be regarded as a generalization of realcompactness provided that, and this is to be explicitely remarked, there exists a measurable cardinal number (see [5] chap. 12).

Now we return to a completely regular space X, without any $\mathfrak{a}$-completeness assumption, and we consider the space $\upsilon_{\mathfrak{a}}(X)$ and the injection j as before.

Then:

3. *If f is a continuous map of X into an $\mathfrak{a}$-complete space Y then there exists a continuous map $\bar{f}$ of $\upsilon_{\mathfrak{a}}(X)$ into Y such that its restriction to $j(X)$ is exactly $f \circ g$ where g is the inverse map of the injection j.*

This property 3 together with properties 1 and 2 of $\upsilon_{\mathfrak{a}}(X)$ and j, determines $\upsilon_{\mathfrak{a}}(X)$ uniquely up to a homeomorphism. Therefore, taking $\mathfrak{a} = \text{card}(\mathbf{N})$ and recalling what we stated before, we obtain the well-known Hewitt extension of X as a dense subset of a Q-space (realcompactification of X, according to [5]) commonly denoted by $\upsilon(X)$.

Further, one must notice that X is an $\mathfrak{a}$-complete space if and only if it is (topologically) identical to $\upsilon_{\mathfrak{a}}(X)$.

Other results may be obtained, which may be considered to be generalizations of analogous results already known for Q-spaces and Hewitt extensions, under the proviso on measurable cardinals mentioned above (see [2] and [3]).

Before closing this paper it must be remarked that the present writer, while attending the Topological Symposium in Prague, has been informed by J. R. Isbell that some results of the same kind have been obtained by that author jointly with S. Ginsburg in [6].

Also as stated in the abstract, the writer does not know whether relations exist between $\mathfrak{a}$-complete spaces and E-compact spaces in the sense of R. Engelking and S. Mrówka (cf. [4]), although similarities exist.

All the results of this paper will be found in full detail, with some more information about $\mathfrak{a}$-completeness, in [2] and in a forthcoming paper [3].

References

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