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SOME APPLICATIONS OF COMPACTNESS IN HARMONIC ANALYSIS

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The discovery in 1922 of the concept of compact Hausdorff space (бикомпактное Хаусдорфово пространство) by P. S. ALEKSANDROV and P. S. URYSON [1] is a landmark in contemporary abstract analysis. Of course the idea of compactness did not begin with this memoir: notions and techniques involving compactness had been used with great effectiveness for many decades prior to 1922; but the final formulation of the concept of compactness, so elegant in its simplicity and so far-reaching in its applications, is the work of Aleksandrov and Uryson.

Two comments on axiomatics may be in order. First, for the purposes of contemporary analysis, compactness has little interest without Hausdorff separation. Real- and complex-valued continuous functions are essential for the work of the analyst, and these (barring constants) may well be absent in a non-Hausdorff compact space. Thus a minimal infinite T_1-space (a subset is closed if and only if it is finite or the entire space) is compact but is interesting to analysts only as a curiosity.[1]) Second, the many generalizations of compactness that have been put forward in recent years may in the future prove valuable for analysis. Thus Lindelöf (finally $\aleph_1$-compact) spaces are frequently useful already; and other notions of this genre may well be used by future workers. Nevertheless, definite limitations exist in the usefulness of non-compact (or non locally compact) spaces, as we will show.

The purpose of this essay is to demonstrate the central rôle of compactness in harmonic analysis. Two theorems from functional analysis, however, are so important to harmonic analysis, and so clearly illustrate the importance of compactness, that they should be cited.

The first of these is the Stone-Weierstrass approximation theorem [1]. Let X be a compact Hausdorff space, and let $\mathfrak{C}(X)$ be the algebra of all *complex*-valued continuous functions on X, where addition and multiplication are as usual pointwise, and where $\|f\|_u = \max\{|f(x)| : x \in X\}$ for $f \in \mathfrak{C}(X)$. Let $\mathfrak{S}$ be a subalgebra of $\mathfrak{C}(X)$ that: (a) separates points of X; (b) is closed under complex conjugation; (c) has the property that for all $x \in X$, there is a $\varphi \in \mathfrak{S}$ such that $\varphi(x) \neq 0$. Then the subalgebra $\mathfrak{S}$ is dense in the topology of $\mathfrak{C}(X)$ induced by the norm $\|\ \|_u$. A proof of a special case, immediately adaptable to the general case, appears in [12]. It would be hard to overstate the

[1]) The closed unit disk in the complex plane receives this topology as the maximal ideal space of a certain Banach algebra of analytic functions: see I. M. GEL'FAND and G. E. ŠILOV [5].

importance of the Stone-Weierstrass theorem. We cite a few applications: to the proof of Fubini's theorem in one of its forms; to computing characters of compact Abelian groups;[2]) to computing irreducible unitary representations of compact non-Abelian groups. The Stone-Weierstrass theorem exhibits a complete dichotomy between the compact and noncompact situations. If X is *any* noncompact, nonvoid, completely regular space, then one can find a subalgebra $\mathfrak{S}$ of $\mathfrak{C}(X)$ satisfying the hypotheses of the Stone-Weierstrass theorem that is not dense in the uniform topology of $\mathfrak{C}(X)$. This is shown in [10].

Our second classical theorem from functional analysis is F. Riesz's representation theorem. Let X now be a *locally* compact Hausdorff space; let $\mathfrak{C}_{00}(X)$ denote the space of all complex-valued continuous functions on X each of which vanishes outside of some compact subset of X (this compact set depending upon the function); let $\mathfrak{C}^r_{00}(X)$ denote the real-valued functions in $\mathfrak{C}_{00}(X)$; and $\mathfrak{C}^+_{00}(X)$ the nonnegative functions in $\mathfrak{C}^r_{00}(X)$. Let I be any linear functional on $\mathfrak{C}_{00}(X)$ that assumes nonnegative real values for all functions in $\mathfrak{C}^+_{00}(X)$.[3]) Then there exists a set-function ι defined for all subsets of X having the following properties:

(1) $0 \leqq \iota(A) \leqq \infty$ for all $A \subset X$;

(2) if A^- is compact, then $\iota(A)$ is finite;

(3) ι is countably additive on a σ-algebra $\mathscr{S}$ of subsets of X that contains all closed sets;

(4) for all $f \in \mathfrak{C}_{00}(X)$, the identity

$$I(f) = \int_X f(x)\,\mathrm{d}\iota(x)$$

obtains.

Thus the functional I is representable as the integral with respect to a *countably* additive measure. The rôle played by countable additivity in integration theory is vital. Without it, Fubini's theorem and Lebesgue's theorem on dominated convergence fail utterly. A close examination of the proof of Riesz's representation theorem shows that compactness of the sets $\{x \in X : f(x) \neq 0\}^-$ is the key to proving that ι is countably additive on $\mathscr{S}$ (see [14], § 11). It is not clear what to say of Riesz's theorem for non locally compact X, since (as in the case of the rational numbers with their usual topology) 0 may be the only function in $\mathfrak{C}_{00}(X)$. However, we can consider a linear functional I on $\mathfrak{C}(X)$ that is nonnegative and real on $\mathfrak{C}^+(X)$, where X is any completely regular space and $\mathfrak{C}(X)$ is the space of all bounded continuous complex-valued functions on X. It is an easy matter to prove that

$$I(f) = \int_X f(x)\,\mathrm{d}\iota(x) \quad \text{for} \quad f \in \mathfrak{C}(X),$$

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[2]) If G is a compact Abelian group and $\mathbf{Y}$ is a subgroup of the character group $\mathbf{X}$ of G such that $\mathbf{Y}$ separates points of G, then $\mathbf{Y} = \mathbf{X}$.

[3]) Note that we suppose no continuity property for I; the condition $I(\mathfrak{C}^+_{00}(X)) \subset [0, \infty[$ is a replacement for continuity.

where now ι is a nonnegative set function on all subsets of X that is *finitely* additive on an algebra of subsets of X that contains all sets

$$\{x \in X : f(x) = 0\} \quad [f \in \mathfrak{C}(X)] .$$

Glicksberg [6] has shown that every such ι is countably additive if and only if X is pseudo-compact.[4])

We pass to some of the applications of compactness in harmonic analysis proper. We shall see that compactness enters not only as an essential hypothesis in many situations, but also as parts of definitions, and as a technique in proving existence theorems.

Abstract harmonic analysis as we know it today could not exist without Haar measure, which we will now describe. Let G be a group and f any function on G. For a, b in G, we denote by ${}_af, f_b$ and ${}_af_b$, respectively, the functions $x \to f(ax)$, $x \to f(xb)$, $x \to f(axb)$, on G. Now suppose that G is a locally compact T_0 group.[5])

Then there exists a linear functional I on $\mathfrak{C}_{00}(G)$ with the following properties:

(i) $I(f)$ is real and nonnegative for $f \in \mathfrak{C}_{00}^{+}(G)$;
(ii) $I({}_af) = I(f)$ for all $a \in G$ and $f \in \mathfrak{C}_{00}(G)$;
(iii) $I \neq 0$.

Such a functional is called a *left Haar integral* on $\mathfrak{C}_{00}(G)$, and the measure λ corresponding thereto by F. Riesz's theorem is called a *left Haar measure.* (It is easy to show that $I(f)$ is strictly positive for $f \neq 0, f \in \mathfrak{C}_{00}^{+}(G)$.) A. Weil's original proof of the existence of a Haar integral [18] made use of compactness in the form of Tihonov's theorem; but H. Cartan [2] shortly after the publication of [18] gave a strictly constructive proof of the existence and uniqueness (up to a multiplicative constant, naturally) of the left Haar integral. Compactness enters in Cartan's proof only in producing a certain partition of unity and in establishing some elementary inequalities. Nevertheless, local compactness is "nearly" essential for proving the existence of Haar measure. If a group G admits an invariant measure and if a certain technical restriction holds, then G is a subgroup of a locally compact group $\tilde{G}$, and G in a certain sense is "large" in $\tilde{G}$. The details are given in P. R. Halmos [9]. It should be pointed out that a finitely additive invariant measure exists on every locally bounded T_0 group (a topological group is locally bounded if it has a neighborhood V of the identity such that a finite number of translates of an arbitrary neighborhood of the identity cover V). This fact was proved by A. A. Markov [15].

The theory of almost periodic functions provides another excellent illustration of the uses of compactness. Consider any T_0 topological group G and any function $f \in \mathfrak{C}(G)$. For $a \in G$, let D_af be the following function on $G \times G : (x, y) \to f(xay)$.

[4]) We recall that a completely regular space X is said to be pseudo-compact if every continuous real-valued function on X is bounded. Such spaces need not be compact: see [11].

[5]) It is well known that a T_0 group is completely regular and that a locally compact T_0 group is normal.

It is elementary, although not completely trivial, to show that the following assertions are equivalent: $\{{}_af : a \in G\}^-$ is compact in $\mathfrak{C}(G)$; $\{f_b : b \in G\}^-$ is compact in $\mathfrak{C}(G)$; $\{{}_af_b : a, b \in G\}^-$ is compact in $\mathfrak{C}(G)$; $\{D_af : a \in G\}$ is compact in $\mathfrak{C}(G \times G)$. (In all cases we use the uniform topology in $\mathfrak{C}$.) A function satisfying one and hence all of these properties is called *almost periodic*. Here compactness (ordinary sequential compactness, it is true, since $\mathfrak{C}(G)$ is a metric space) is a part of the definition. There is a complete theory of almost periodic functions. The space $\mathfrak{A}(G)$ of almost periodic functions on G admits a unique nonnegative left invariant mean value (whose existence is proved by wholly elementary arguments), which is right and inversion invariant. Functions in $\mathfrak{A}(G)$ are uniform limits of linear combinations of coefficients of finite-dimensional continuous unitary representations of G; and so on.

A promising generalization of almost periodicity was advanced a few years ago by W. F. EBERLEIN [3]. Topologize $\mathfrak{C}(G)$ not with the uniform topology but with the weak topology based on linear functionals in the conjugate space $\mathfrak{C}^*(G)$. Say that a function $f \in \mathfrak{C}(G)$ *is weakly almost periodic* if $\{{}_af : a \in G\}^-$ or $\{f_b : b \in G\}^-$ is compact in the weak topology for $\mathfrak{C}(G)$ (the two conditions are equivalent). One may then ask if the space of weakly almost periodic functions for a locally compact G admits an invariant mean value. For Abelian G, it is trivial that there is such a mean, since the space of all bounded functions admits an invariant mean in this case. For non-Abelian G, the problem seems to be unsolved. (For partial results, see Glicksberg and de Leeuw [7].)

Our third illustration of the uses of compactness in abstract harmonic analysis is the proof of the famous theorem of I. M. GEL'FAND and D. A. RAĬKOV [4]. (See also [8].) Let G be a topological group. A *continuous unitary representation* of G is a mapping $x \to U_x$ of G into the group of unitary operators on some Hilbert space $\mathscr{H}$ such that $U_{xy} = U_xU_y$ for all $x, y \in G$ and such that $x \to \langle U_x\xi, \eta\rangle$ is a continuous function on G for all $\xi, \eta \in \mathscr{H}$. A unitary representation U is *irreducible* if there is no closed subspace $\mathscr{S}$ of $\mathscr{H}$ distinct from $\{0\}$ and $\mathscr{H}$ such that $U_x(\mathscr{S}) \subset \mathscr{S}$ for all $x \in G$. The Gel'fand-Raĭkov theorem asserts that if G is locally compact, then for every x in G different from the identity, there is an irreducible unitary representation U of G such that U_x is not the identity operator. This theorem implies at once the Peter-Weyl theorem and the fact that a locally compact Abelian group admits sufficiently many continuous characters. In addition it has inspired an immense amount of research on computing the irreducible unitary representations (for the most part infinite-dimensional) of the classical groups.

The proof of the Gel'fand-Raĭkov theorem is somewhat technical. There are two versions of it, one based on positive-definite functions on G, another based on the following considerations. Let $\mathfrak{L}_1(G)$ denote the Banach space of all Borel (let us say) measurable complex-valued functions f on the locally compact group G for which

$$\|f\|_1 = \int_G |f(x)| \, d\lambda(x) < \infty ,$$

where λ denotes a left Haar measure on G. For f, g in $\mathfrak{L}_1(G)$, define the function $f * g$ (convolution, Faltung, or свертка) by

$$f * g(x) = \int_G f(xy)\, g(y^{-1})\, \mathrm{d}\lambda(y), \quad \text{for} \quad x \in G .$$

It can be shown that $f * g(x)$ exists and is a complex number for λ-almost all $x \in G$, that the function $f * g$ defined in this fashion is in $\mathfrak{L}_1(G)$, and that the inequality

$$\|f * g\|_1 \leqq \|f\|_1 \cdot \|g\|_1$$

obtains for all f, g in $\mathfrak{L}_1(G)$. With pointwise linear operations and multiplication defined as convolution, $\mathfrak{L}_1(G)$ is thus a (complex) Banach algebra.

In addition, the algebra $\mathfrak{L}_1(G)$ admits an involution. For $x \in G$ let

$$\Delta(x) = \int_G f_{x^{-1}}(y)\, \mathrm{d}\lambda(y) \Big/ \int_G f(y)\, \mathrm{d}\lambda(y) ,$$

where f is any nonzero function in $\mathfrak{C}_{00}^{+}(G)$; Δ is continuous and positive and satisfies the relation $\Delta(xy) = \Delta(x)\,\Delta(y)$ for all x, y in G. (The fact that Δ depends on x alone follows from the uniqueness of left Haar measure.) Now for $f \in \mathfrak{L}_1(G)$, let $f^{\sim}$ be defined by

$$f^{\sim}(x) = \overline{f(x^{-1})}\, \frac{1}{\Delta(x)} .$$

The mapping $f \to f^{\sim}$ is an involution on $\mathfrak{L}_1(G)$.

A linear functional p on $\mathfrak{L}_1(G)$ is called *positive* if $p(f^{\sim} * f)$ is real and nonnegative for all $f \in \mathfrak{L}_1(G)$. A proof of the Gel'fand-Raĭkov theorem can be given that depends upon a close analysis of positive functionals on $\mathfrak{L}_1(G)$, their connection with representations of $\mathfrak{L}_1(G)$ by operators on Hilbert spaces, and the connection of these with continuous unitary representations of G itself. The subspace $\mathfrak{H}$ of $\mathfrak{L}_1(G)$ consisting of all functions such that $f = f^{\sim}$ is a real Banach space. The set P of all positive functionals p on $\mathfrak{L}_1(G)$ such that $p(f^{\sim}) = p(f)$ and $|p(f)| \leqq p(f^{\sim} * f)^{\frac{1}{2}}$ for all $f \in \mathfrak{L}_1(G)$ is a compact convex subset of the (real!) conjugate space $\mathfrak{H}^*$. By the Kreĭn-Mil'man theorem, P is the ($*$-weak) closure of the convex hull of its own extreme points. The extreme points of P correspond to irreducible representations of $\mathfrak{L}_1(G)$, and so in a certain sense every representation of $\mathfrak{L}_1(G)$ can be approximated by irreducible representations. Finally, the mapping $\varphi \to f * \varphi = T_f\varphi$ of $\mathfrak{L}_2(G)$ into itself is a bounded linear operator for all $f \in \mathfrak{L}_1(G)$ and the mapping $f \to T_f$ is a faithful representation of $\mathfrak{L}_1(G)$ by operators on the Hilbert space $\mathfrak{L}_2(G)$. Note too that $(T_f)^{\sim} = T_{f^{\sim}}$, where $T^{\sim}$ denotes the adjoint operator to T.

The facts just outlined give a proof of the Gel'fand-Raĭkov theorem. The crux of the proof is of course the Kreĭn-Mil'man theorem: and this theorem depends wholly upon compactness.

For a compact group G, the function spaces $\mathfrak{C}(G)$, $\mathfrak{L}_2(G)$, and $\mathfrak{L}_1(G)$ are Banach algebras under convolution. Their structure is completely known: all maximal ideals and closed ideals in these algebras have been identified. Here one may say that compactness has registered another success. For noncompact locally compact G, very little is known of the detailed structure of $\mathfrak{L}_1(G)$ ($\mathfrak{C}(G)$ and $\mathfrak{L}_2(G)$ are not algebras in this case). Another Banach algebra can be defined for every locally compact G. Let $\mathfrak{C}_0(G)$ be the Banach space of all continuous complex-valued functions f on G such that for every $\varepsilon > 0$, the set $\{x : x \in G, |f(x)| \geqq \varepsilon\}$ is compact. Let $\mathcal{M}(G)$ denote the conjugate space of $\mathfrak{C}_0(G)$. It is convenient to use F. Riesz's theorem to represent elements of $\mathcal{M}(G)$ as measures. Then for μ, ν in $\mathcal{M}(G)$ and $f \in \mathfrak{C}_0(G)$, let

$$\mu * \nu(f) = \iint\limits_{GG} f(xy)\, \mathrm{d}\nu(y)\, \mathrm{d}\mu(x)\,.$$

This definition is an extension of the definition of convolution for functions in $\mathfrak{L}_1(G)$ (regard functions in $\mathfrak{L}_1(G)$ as measures absolutely continuous with respect to left Haar measure). The algebra $\mathcal{M}(G)$ has been extensively studied (for example, see [16] and [13]), but its detailed structure remains a nearly complete mystery, even for the simplest compact infinite groups. A complete analysis of $\mathcal{M}(G)$ even for compact Abelian groups G would be of the greatest interest. The analysis of $\mathcal{M}(G)$ for general G and of $\mathfrak{L}_1(G)$ for noncompact G would seem to be one of the most important problems now open in harmonic analysis.

In conclusion, we remark that most of the matters discussed in this essay are treated in detail in a forthcoming book [14].

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