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THE SPACE OF MINIMAL PRIME IDEALS OF A COMMUTATIVE RING

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1. Introduction. Our interest in the space of minimal prime ideals of a commutative ring arises from the special features of this space in case the ring is $C(X)$, the ring of all continuous real-valued functions on a topological space X. For instance, if X is the one-point compactification of a countable discrete space N, then the space of minimal prime ideals of $C(X)$ is homeomorphic with βN, the Stone-Čech compactification of N. This was pointed out by C. W. KOHLS [4], who initiated the study of minimal prime ideals in rings $C(X)$.

It should be noted at the outset that a *minimal* prime ideal means a prime ideal that contains no smaller prime ideal. Thus, in an integral domain, the only minimal prime ideals is (0). The following lemma provides a key tool for the study of such ideals in an arbitrary ring, which we always assume to be commutative.

1.1. Lemma. *A proper prime ideal P of a ring A is minimal if and only if for each $x \in P$ there exists $a \in A \sim P$ such that ax is nilpotent.*

It is easy to see that the stated condition is sufficient for minimality of P. To prove necessity, one assumes that the condition is violated and uses a standard argument involving Zorn's lemma to construct a prime ideal contained properly in P.

2. *The space $\mathscr{P}(A)$.* Let $\mathscr{P}$, or more precisely $\mathscr{P}(A)$, denote the set of all minimal prime ideals in a ring A. The *hull* of a set $S \subset A$ is

$$h(S) = \{P \in \mathscr{P} : S \subset P\} .$$

The *kernel* of a set $\mathscr{S} \subset \mathscr{P}$ is

$$k(\mathscr{S}) = \bigcap \{P : P \in \mathscr{S}\} .$$

A topology is defined in $\mathscr{P}$ by means of a closure operation: The *closure* of $\mathscr{S}$ is the set $h\,k(\mathscr{S})$. Evidently, the family of sets $\{h(a) : a \in A\}$ is a base for the closed sets in $\mathscr{P}$.

2.1. Theorem. *Let I be an ideal of A. The mapping τ defined by*

$$\tau(P) = P \cap I , \quad P \in h(I)$$

is a homeomorphism of the subspace $h(I)$ of $\mathscr{P}(A)$ onto a subspace of $\mathscr{P}(A/I)$.

The proof of this theorem is straightforward. In case I is the ideal of all nilpotents in A, then $h(I)$ is all of $\mathscr{P}(A)$ and the image under τ is all of $\mathscr{P}(A/I)$. Consequently, we lose no generality in studying topological properties of $\mathscr{P}(A)$ if we assume that A has no nonzero nilpotent.

For any $S \subset A$, we denote the *annihilator of S by* $\mathfrak{A}(S)$:

$$\mathfrak{A}(S) = [a \in A : as = 0 \text{ for all } s \in S] .$$

2.2. Theorem. *For any element a in a ring without nonzero nilpotent,* $h(\mathfrak{A}(a)) = \mathscr{P} \sim h(a)$. *Thus, besides being closed by definition, the sets* $h(a)$ *and* $h(\mathfrak{A}(a))$ *are open.*

This theorem follows directly from Lemma 1.1 which, for rings without nonzero nilpotent, simply says: a prime ideal P is minimal if and only if for all $x \in P$, $\mathfrak{A}(x) \not\subset P$.

2.3. Corollary. $\mathscr{P}$ *is a Hausdorff space with a base of open-and-closed sets.*

2.4. Corollary. *An element in a ring without nonzero nilpotent belongs to some minimal prime ideal if and only if it is a divisor of zero.*

Additional useful properties of annihilators and hulls of elements are:

2.5. Lemma. *For all x, y, z in a ring without nonzero nilpotent,*

(i) $h(x) = h(\mathfrak{A}\,\mathfrak{A}(x))$;

(ii) $\mathfrak{A}\,\mathfrak{A}(xy) = \mathfrak{A}\,\mathfrak{A}(x) \cap \mathfrak{A}\,\mathfrak{A}(y)$;

(iii) $\mathfrak{A}(z) = \mathfrak{A}(x) \cap \mathfrak{A}(y)$ *if and only if* $h(z) = h(x) \cap h(y)$;

(iv) $\mathfrak{A}\,\mathfrak{A}(y) = \mathfrak{A}(x)$ *if and only if* $h(y) = h(\mathfrak{A}(x))$.

3. *Compactness of* $\mathscr{P}$. A striking difference between the space $\mathscr{P}(\mathfrak{A})$ and more familiar spaces of ideals of a ring A (see, e. g. [1]) is that compactness of $\mathscr{P}(A)$ is wholly unrelated to the presence of a unity in A. Instead, compactness of $\mathscr{P}(A)$ hinges upon the existence of a kind of complement in the sense that for each element x of A there shall exist $x' \in A$ such that $\mathfrak{A}\,\mathfrak{A}(x') = \mathfrak{A}(x)$. This condition is sufficient for compactness; we have been able to prove that it is necessary only under the additional restriction stated next.

3.1. Definition. *A ring A is said to satisfy the* annihilator condition *(or is an a. c. ring) if A has no nonzero nilpotent and for every* $x, y \in A$, *there exists* $z \in A$ *such that* $\mathfrak{A}(z) = \mathfrak{A}(x) \cap \mathfrak{A}(y)$.

It is difficult to find a ring without nonzero nilpotent that is not *a. c. Professor* Harley Flanders *provided the following example which, moreover, has a unity.*

3.2. Example. Let K be an algebraically closed field and $\Lambda = K \times K \times K$. In the ring K^Λ of all K-valued functions on Λ, let F be the subring of functions that are 0 except on a finite subset of Λ. Define $x, y \in K^\Lambda$ by

$$x(a, b, c) = a\,, \quad y(a, b, c) = b\,, \quad ((a, b, c) \in \Lambda)\,,$$

and let A be the smallest subring of K^Λ that contains F, x, y, and the constants. In the ring A, $\mathfrak{A}(x) \cap \mathfrak{A}(y)$ is not the annihilator of any single element.

3.3. Theorem. *The following conditions on a ring A without nonzero nilpotent are equivalent:*

(a) *A is a. c. and* $\mathscr{P}(A)$ *is compact.*

(b) *For each* $x \in A$ *there exists* $x' \in A$ *such that* $\mathfrak{A}\,\mathfrak{A}(x') = \mathfrak{A}(x)$.

Proof. (a) implies (b). For a given $x \in A$, the existence of x' will follow from compactness of $h(x)$ only. By Theorem 2.2,

$$\bigcap\{h(y) \cap h(x) : y \in \mathfrak{A}(x)\} = h(\mathfrak{A}(x)) \cap h(x) = \emptyset .$$

Hence, there exist $y_1, \ldots, y_n \in \mathfrak{A}(x)$ such that

$$h(y_1) \cap \ldots \cap h(y_n) \cap h(x) = \emptyset ,$$

which implies $h(y_1) \cap \ldots \cap h(y_n) = h(\mathfrak{A}(x))$. Since A is a. c., there exists $x' \in A$ such that $\mathfrak{A}(x') = \mathfrak{A}(y_1) \cap \ldots \cap \mathfrak{A}(y_n)$. By Lemma 2.5, we have, $h(x') = h(\mathfrak{A}(x))$ and hence $\mathfrak{A}\,\mathfrak{A}(x') = \mathfrak{A}(x)$.

Now assume (b). That A is a. c. follows from Lemma 2.5 (ii), if we set $z = (x'y')'$. To prove that $\mathscr{P}(A)$ is compact, we need

3.4. Lemma. *Let A satisfy* (b). *An ideal I in A is contained in some minimal prime ideal of A if (and only if) every member of I is a divisor of* 0. *In particular, a prime ideal in A is minimal if and only if each of its members is a divisor of* 0. (*Cf. Corollary* 2.4).

To prove this lemma, we use Zorn's lemma to embed I in a prime ideal P each of whose members is a divisor of 0. Suppose that P is not minimal. Then there exists $x \in P$ such that $\mathfrak{A}(x) \subset P$. The element x' such that $\mathfrak{A}\,\mathfrak{A}(x') = \mathfrak{A}(x)$ clearly belongs to $\mathfrak{A}(x)$, and so $x' \in P$. Hence $x + x' \in P$. But $h(x') = h(\mathfrak{A}(x)) = \mathscr{P} \sim h(x)$, that is, every minimal prime ideal contains exactly one of x and x'. Thus, $h(x + x') = \emptyset$, so by Corollary 2.4, $x + x'$ is not a divisor of 0. This contradicts $x + x' \in P$.

To complete the proof of the theorem, let $\{h(x_\alpha)\}$ be a family of basic closed sets in $\mathscr{P}$ with empty intersection. If I is the ideal generated by $\{x_\alpha\}$, then $h(I) = \bigcap h(x_\alpha) = \emptyset$. By the lemma, I contains a nondivisor of 0, say e. Then there exist $x_{\alpha_1}, \ldots, x_{\alpha_n}$ and $a_1, \ldots, a_n$ in A such that $e = \sum_{i=1}^{n} a_i x_{\alpha_1}$, and we have

$$\bigcap_{i=1}^{n} h(x_{\alpha_i}) \subset h(e) = \emptyset .$$

4. *Countable compactness and basic disconnectivity of $\mathscr{P}$.*

4.1. Definition. *A ring A is said to satisfy the countable annihilator condition (or is a c. a. c. ring) if A has no nonzero nilpotent and for each sequence. $\{x_n\}$ in A, there exists $x \in A$ such that $\mathfrak{A}(x) = \bigcap_{n=1}^{\infty} \mathfrak{A}(x_n)$.*

Obviously, every c. a. c. ring is a. c. Any ring $C(T)$ is a c. a. c. ring, as follows: $x(t) = \sum_{n=1}^{\infty} 2^{-n} \min(|x_n(t)|, 1)$ for all $t \in T$.

4.2. Lemma. *If B is a set in any ring A, then in the space $\mathscr{P}(A)$, the closure of $\bigcup\{h(\mathfrak{A}(b)) : b \in B\}$ is $h(\mathfrak{A}(B))$.*

4.3. Corollary. *If for every $B \subset A$, there exists $x \in A$ such that $\mathfrak{A}(B) = \mathfrak{A}(x)$, then $\mathscr{P}(A)$ is extremally disconnected.*

The hypothesis of this corollary is an obvious strengthening of the countable annihilator condition. If only c. a. c. is assumed, the most that might be expected of $\mathscr{P}(A)$ is that it be basically disconnected – a countable analogue of extremal disconnectivity – which is defined in [2] as follows: a space X is *basically disconnected* if every zero-set in X has a closed interior. However, c. a. c. by itself is not sufficient to make $\mathscr{P}(A)$ basically disconnected. Our most general result in this direction is

4.4. Theorem. *If A is a c. a. c. ring and $\mathscr{P}(A)$ is locally compact, then $\mathscr{P}(A)$ is basically disconnected.*

Next, we have a property of all c. a. c. rings.

4.5. Theorem. *If A is a c. a. c. ring, then $\mathscr{P}(A)$ is countably compact.*

A cluster point for an arbitrary sequence $\{P_n\}$ in such a $\mathscr{P}(A)$ is constructed as follows: Let $\mathscr{U}$ be an ultrafilter on the integers with void intersection. For each $x \in A$, let $E(x) = \{n : x \in P_n\}$. Then the set $\{x \in A : E(x) \in \mathscr{U}\}$ is a minimal prime ideal of A and is a cluster point of $\{P_n\}$. The countable annihilator condition is used only in the proof of minimality.

5. *Minimal prime ideals of* Φ-algebras. An archimedean lattice-ordered algebra over the real field with a unity element 1 that is also a weak order unit is called a *Φ-algebra*. As was shown in [3], a Φ-algebra is a natural generalization of a ring $C(X)$, especially if one is concerned with spaces of ideals. If A is a Φ-algebra, then the space $\mathscr{M}(A)$ of all maximal l-ideals (an l-ideal I is a ring ideal such that $b \in I$ and $|a| \leqq |b|$ implies $a \in I$) of A is a compact Hausdorff space. The set

$$A^* = \{a \in A : |a| \leqq \lambda \,.\, 1 \text{ for some real } \lambda\}$$

of bounded elements of A is also a Φ-algebra, and $\mathscr{M}(A^*)$ is homeomorphic with $\mathscr{M}(A)$. It is also true that $\mathscr{P}(A^*)$ is homeomorphic with $\mathscr{P}(A)$. Any Φ-algebra A is isomorphic with a Φ-algebra of extended real-valued, continuous functions on the space $\mathscr{M}(A)$ that are real-valued on a dense subset of $\mathscr{M}(A)$. The isomorphism carries A^* onto a subalgebra of $C(\mathscr{M}(A))$.

Each minimal prime ideal of A is contained in a unique maximal l-ideal. Thus, there is a mapping ι of $\mathscr{P}(A)$ *onto* $\mathscr{M}(A)$, which is automatically continuous. We shall state the properties of this mapping for the case $A = C(X)$, where X is a compact Hausdorff space; in view of the remarks of the preceding paragraph, this entails little loss of generality. Also, we use the well-known homeomorphism between X and $\mathscr{M}(C(X))$ and regard ι as a mapping of $\mathscr{P}$ onto X – for $P \in \mathscr{P}$, $\iota(P)$ is the unique point in X where all functions in P vanish.

5.1. Theorem. (a) *ι is one-one if and only if X is an F-space* [2, p. 208].

(b) *ι is a homeomorphism if and only if X is basically disconnected.*

(c) *In case X is an F-space, $\mathscr{P}$ is compact if and only if X is basically disconnected.*

Any $C(X)$ is an a. c. ring, in fact, a c. a. c. ring. Condition (b) of Theorem 3.3 is therefore necessary and sufficient in order that $\mathscr{P}(C(X))$ be compact. In terms of the behavior of functions, this condition says:

(b') For each $f \in C(X)$, there exists $f' \in C(X)$ such that

$$Z(f) \cup Z(f') = X \quad \text{and} \quad \operatorname{int}[Z(f) \cap Z(f')] = \emptyset .$$

($Z(f)$ denotes the zero-set of f.) A more easily verified sufficient, but not necessary, condition for $\mathscr{P}(C(X))$ to be compact is given next.

5.2. Theorem. *If, for every $f \in C(X)$, the support of f is a zero-set, in particular if X is perfectly normal, then $\mathscr{P}(C(X))$ is compact.*

We conclude with some illuminating examples.

5.3. A space Γ such that $\mathscr{P}(C(\Gamma))$ is locally compact, but not compact. Let W^* be the totally ordered space of ordinal numbers less than or equal to the first uncountable ordinal, ω_1. Γ is the quotient space of W^* obtained by identifying ω_0 with ω_1. A function $f \in C(\Gamma)$ for which (b') fails may be defined as follows: $f(n) = 1/n$ if $0 < n < \omega_0, f(\gamma) = 0$ otherwise. Hence $\mathscr{P}(C(\Gamma))$ is not compact. It is locally compact, however.

5.4. A space X such that $\mathscr{P}(C(X))$ is compact but for which the hypothesis of Theorem 5.2 is not satisfied. Let N^* denote the totally ordered space of ordinals less than or equal to ω_0. With Γ as in 5.3, X is the complement in $\Gamma \times N^*$ of the set $\{(m, n) : m < \omega_0, n < \omega_0\}$.

5.5. A space X such that no open set in $\mathscr{P}(C(X))$ has compact closure. Let $X = \beta N \sim N$, where N is a countable discrete space. We do not know whether $\mathscr{P}(C(X))$ is basically disconnected.

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