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On dimension and metrization

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ON DIMENSION AND METRIZATION[1])

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As is well known, it follows from results of E. MARCZEWSKI [4] that

A separable metric space R has dim $\leqq n$ *if and only if we can introduce a topology-preserving metric* ϱ *such that almost all of the spherical neighborhoods of any point of R have boundaries of* dim $\leqq n - 1$.

For general metric spaces we proved [6] the following theorems:

A metric space R has dim $\leqq n$ *if and only if we can introduce a topology-preserving metric into R such that the spherical neighborhoods* $S_{1/i}(p)$, $i = 1, 2, \dots$ *of any point p of R have boundaries of* dim $\leqq n - 1$ *and such that* $\{S_{1/i}(p) \mid p \in R\}$ *is closure preserving for every i.*

Let $S_{1/i}(F) = \cup \{S_{1/i}(p) \mid p \in F\}$; $B(S)$ *denotes the boundary of S. A metric space R has* dim $\leqq n$ *if and only if we can introduce a topology-preserving metric into R such that*

$$\dim B(S_{1/i}(F)) \leqq n - 1\,, \quad i = 1, 2, \dots$$

for every closed subset F of R.

In the above theorem $S_{1/i}(p)$ denotes the spherical neighborhood $\{q \mid \varrho(p, q) < 1/i\}$. A collection $\mathfrak{A}$ of subsets of R is called closure preserving if $\bigcup\{\bar{A} \mid A \in \mathfrak{A}'\} = \overline{\bigcup\{A \mid A \in \mathfrak{A}'\}}$ for any subset $\mathfrak{A}'$ of $\mathfrak{A}$; dim R denotes the covering dimension of R, but since M. KATĚTOV [3] and K. MORITA [5] have proved dim R = Ind R for any metric space R, in metric spaces we do not distinguish between them.

We expected there would be many difficulties to introduce into n-dimensional general metric spaces a metric in which much more spherical neighborhoods have boundaries of dim $\leqq n - 1$, because the p-dimensional measure does not work in general metric spaces though it played the leading role in Marczewski's theory for separable metric spaces.

We, however, have succeeded to prove without measure the following theorem and its corollaries quite recently.

Theorem 1. *A metric space R has* dim $\leqq n$ *if and only if we can introduce a topology-preserving metric* ϱ *into R such that all spherical neighborhoods* $S_\varepsilon(p)$ *of any point P of R have boundaries of* dim $\leqq n - 1$ *and such that* $\{S_\varepsilon(p) \mid p \in R\}$ *is closure preserving for every real number* ε.

[1]) The content of this note will be published in a more extended form in some other place.

This metric is rather peculiar in view that the usual metric of Euclidean space does not satisfy the closure preserving condition. But the metric in the following corollary will be more reasonable.

Corollary 1. *A metric space R has* dim $\leqq n$ *if and only if we can introduce a metric* ϱ *into R such that*

$$\dim B(S_\varepsilon(F)) \leqq n - 1$$

for any real number ε *and for any closed set F of R.*

Now let $C_\varepsilon(p) = \{q \mid \varrho(p, q) = \varepsilon\}$; then $C_\varepsilon(p) = B(S_\varepsilon(p))$ is not always valid. But the metric applied to the proof of the preceding theorem satisfies $C_\varepsilon(p) = B(S_\varepsilon(p))$ for almost all ε or more precisely for all irrational ε and some rational ε. Thus we get the following corollaries, too.

Corollary 2. *A metric space R has* dim $\leqq n$ *if and only if we can introduce a metric* ϱ *into R such that*

$$\dim C_\varepsilon(p) \leqq n - 1$$

for every irrational (or almost all) ε *and for every point p of R and such that* $\{S_\varepsilon(p) \mid p \in R\}$ *is closure preserving for any irrational (or almost all)* ε.

Corollary 3. *A metric space R has* dim $\leqq n$ *if and only if we can introduce a metric* ϱ *into R such that*

$$\dim C_\varepsilon(F) \leqq n - 1$$

for every closed set F of R and for every irrational (or almost all) ε, *where* $C_\varepsilon(F) = \{p \mid \varrho(p, F) = \varepsilon\}$.

The point of proof of this theorem is how to introduce into an n-dimensional metric space R a metric satisfying the conditions. To do it we choose a sequence $\{\mathfrak{U}_i \mid i = 0, 1, 2, \ldots\}$ of open coverings of R such that

1. $\{R\} = \mathfrak{U}_0 > \mathfrak{U}_1^{**} > \mathfrak{U}_1 > \mathfrak{U}_2^{**} > \mathfrak{U}_2 > \mathfrak{U}_3^{**} > \ldots$
2. $\{S(p, \mathfrak{U}_m) \mid m = 0, 1, 2, \ldots\}$ is a neighborhoood basis of each point p of R.
3. $S^2(p, \mathfrak{U}_{m+1}^*)$ intersects at most $n + 1$ sets of $\mathfrak{U}_m$,

where the terminologies about coverings are due to [9].

For integers $m_1, m_2, \ldots, m_k$ with $1 \leqq m_1 < m_2 < \ldots < m_k$ and for $U \in \mathfrak{U}_{m_1}$ we define open sets $S_{m_1,m_2,\ldots,m_k}(U)$ by

$$S_{m_1}(U) = U\,; \quad S_{m_1,\ldots,m_k}(U) = S^2(S_{m_1,\ldots,m_{k-1}}(U), \mathfrak{U}_{m_k})\,, \quad k \geqq 2\,.$$

Then we define open coverings $\mathfrak{S}_{m_1,\ldots,m_k}$ of R by

$$\mathfrak{S}_{m_1} = \mathfrak{U}_{m_1}\,, \quad m_1 \geqq 0\,;$$

$$\mathfrak{S}_{m_1,\ldots,m_k} = \{S_{m_1,\ldots,m_k}(U) \mid U \in \mathfrak{U}_{m_1}\}\,, \quad 1 \leqq m_1 < \ldots < m_k\,, \quad k \geqq 2$$

to define a function $\varrho(x, y)$ over $R \times R$ by

$$\varrho(x, y) = \inf\{2^{-m_1} + 2^{-m_2} + \ldots + 2^{-m_k} \mid y \in S(x, \mathfrak{S}_{m_1,\ldots,m_k})\}\,.$$

We can prove this ϱ is the desired metric.

As a matter of fact, we applied this metric to characterizing dimension of metric spaces in another way [7], [8]. That characterization theorem was simplified in separable cases by J. de Groot [2] as follows.

A separable metric space R has dim $\leqq n$ *if and only if we can introduce a totally bounded metric ϱ into R such that for every $n+3$ points*

$$x, y_1, y_2, \ldots, y_{n+2}$$

in R there is a triplet of indices i, j, k satisfying

$$\varrho(y_i, y_j) \leqq \varrho(x, y_k) \quad (i \neq j).$$

It will be an interesting problem to find a simple condition for n-dimensionality of *general* metric spaces in this direction. In this connection we have unsuccessfully tried to prove the following conjecture.

Let R be a metric space of dim $\leqq n$; then can we introduce into R a metric ϱ such that for every $n+3$ points

$$x, y_1, y_2, \ldots, y_{n+2}$$

in R there is a pair of indices i, j satisfying

$$\varrho(y_i, y_j) \leqq \varrho(x, y_j) \quad (i \neq j).$$

We, however, have got in this try a new n-dimensionality theorem which will have its own interest. To show that theorem we need some terminologies.

Definitions. Two subsets A and B of a space R are called *independent* if $A \not\subset B$, $B \not\subset A$. A collection $\mathfrak{A}$ of subsets is called independent if any two members of $\mathfrak{A}$ are independent. Let $\mathfrak{A}$ be a collection of subsets of R and p a point of R. Then $\text{rank}_p\, \mathfrak{A}$ is the largest number of independent sets of $\mathfrak{A}$ which contain p. Moreover

$$\text{rank}\, \mathfrak{A} = \max \{\text{rank}_p\, \mathfrak{A} \mid p \in R\}.$$

It is clear that rank $\mathfrak{A} \leqq$ order $\mathfrak{A}$ for every collection $\mathfrak{A}$ of subsets. Now we can prove the following theorem.

Theorem 2. *A metric space R has* dim $\leqq n$ *if and only if it has an open basis $\mathfrak{A}$ of* rank $\mathfrak{A} \leqq n+1$.

Let us end this note with some problems. Aside from dimension, we do not know whether every metric space R has an open basis $\mathfrak{A}$ of $\text{rank}_p\, \mathfrak{A} < +\infty$ at each point p of R or not. Alexandroff's latest metrization theorem [1] assures us only that every metric space R has an open basis $\mathfrak{A}$ such that any independent subcollection of $\mathfrak{A}$ having a common point p is finite. Conversely, a topological space with an open basis of rank $< +\infty$ is not necessarily metrizable. We can easily give an example that is a regular space with an open basis of rank $= 1$ but not metrizable. Then, what space is the topological space which has an open basis $\mathfrak{A}$ of rank $\mathfrak{A} < +\infty$, and what about the topological space which has an open basis of $\text{rank}_p\, \mathfrak{A} < +\infty$ at every point p of R.

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