CONCERNING THE DIMENSION OF ANR-SETS

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I shall understand here by ANR-sets only compact absolute neighbourhood retracts. These sets constitute a class of spaces which is much more general than the class of all finite polytopes. However, the ANR-sets have topological properties similar in many respects to topological properties of polytopes.

In the present communication I intend to give a simple theorem exhibiting a further analogy between the dimensional properties of ANR-sets and of polytopes.

It is a very elementary fact, that a family of n-dimensional disjoint sets lying in an n-dimensional polytope is at most countable. An analogous statement for arbitrary n-dimensional compacta is not true. For instance, the Cartesian product $Q^n \times C$ of the n-dimensional ball Q^n with the Cantor discontinuum C is an n-dimensional compactum which contains a family of $2^{\aleph_0}$ n-dimensional disjoint balls of the form $Q^n \times (x)$, with $x \in C$.

For ANR-spaces an analogous phenomenon is impossible. In fact, we have the following

Theorem. *Let X ba an ANR-set and let $\{K_\alpha\}$ be a family of n-dimensional ANR-sets lying in X and indexed by α which runs over an uncountable set A. If for every two distinct indices $\alpha, \alpha' \in A$ the dimension of the common part of K_α and $K_{\alpha'}$ is less than n, then the dimension of X is greater than n.*

In order to prove this theorem, let us assume that X is a subset of the Hilbert space H^ω. Since X is an ANR-set

(1) *There exists a neighbourhood U of X in H^ω and a retraction $r: U \to X$ of U to X.*

Since dim K_α is equal to n, there exists in K_α an infinite n-dimensional chain such that its boundary lies in a compactum $B_\alpha \subset K_\alpha$, and there exists a positive number ε_α such that the boundary of this chain is not homologous to zero in the generalized ball

$$Q_\alpha = \underset{x \in K_\alpha}{E} \left[\varrho(x, B) < \varepsilon_\alpha\right].$$

By an infinite chain in K_α we understand here a sequence $\{K_{\alpha,i}\}$ of n-dimensi nal chains lying in K_α, with coefficients belonging to arbitrary Abelian groups, in general depending on i, and with maximal diameter of simplexes converging to zero when i

tends to infinity. By the boundary of this chain we understand the infinite cycle $\{\partial K_{\alpha,i}\}$.

Hence

(2) $\{\partial K_{\alpha,i}\}$ *lies in* $B_\alpha \subset K_\alpha$ *and* $\{\partial K_{\alpha,i}\} \nsim 0$ *in* $Q_\alpha = \underset{x \in K_\alpha}{E} [\varrho(x, B_\alpha) < \varepsilon_\alpha]$.

In general, the positive number ε_α depends on α. However, since α runs over the uncountable set A, there exists an $\varepsilon > 0$ such that $\varepsilon_\alpha > \varepsilon$ for an uncountable set of indices α. Consequently, if we replace A by its suitably chosen subset, we can assume that

(3) $$\varepsilon_\alpha > \varepsilon > 0 \textit{ for every } \alpha \in A .$$

The compacta K_α may be considered as points of the space 2^X consisting of all non-empty subcompacta of X. Since 2^X is compact and since A is uncountable, we infer easily that there exists an index β in A and a sequence $\{\alpha_m\}$ of distinct indices such that

(4) $$\lim K_{\alpha_m} = K \quad \textit{and} \quad \alpha_m \neq \beta \quad \textit{for} \quad m = 1, 2, \ldots$$

Since K_β is an ANR-set, we infer that

(5) *There exists a neighbourhood V of K in H^ω and a retraction s of V to K_β.*

Now we see easily that there exists a positive integer n_0 such that for the index $\gamma = \alpha_{m_0}$ every segment $\overline{x\, s(x)}$ (in H^ω) with $x \in K_\gamma$ lies in $U \cap V$ and that the diameter of the set $r(\overline{x\, s(x)})$ is $< \varepsilon$:

$$\overline{x\, s(x)} \subset U \cap V \quad \textit{and} \quad \delta[r(\overline{x\, s(x)})] < \varepsilon \quad \textit{for every} \quad x \in K_\gamma .$$

Setting

$$f_t(x) = r[(1-t)\, x + t\, s(x)] \quad \text{for every} \quad 0 \leqq t \leqq 1 ,$$

we see easily that the family of functions $\{f_t\}$ is a homotopical deformation of the set K_γ in the space X to the set K_β.

By (2) and (3), there exists in K_γ an infinite n-dimensional chain $\{K_{\gamma,i}\}$ such that the infinite cycle $\{\partial K_{\gamma,i}\}$ lies in a compactum $B_\gamma \subset K_\gamma$ and it is not homologuous to zero in the ball

$$Q_\gamma = \underset{x \in K_\gamma}{E} [\varrho(x, B_\gamma) < \varepsilon] .$$

Let us consider the compactum

$$M = \bigcup_{x \in B_\gamma} (\overline{x\, s(x)}) .$$

Since the diameter of the set $r(\overline{x\, s(x)})$ is smaller than ε and since $r(x) = x \in B_\gamma$, we infer that $r(M) \subset Q_\gamma$.

Evidently $f_t(x) \in r(M) \subset Q_\gamma$ for every point $x \in B_\gamma$. We conclude that there exists in the space X an infinite $(n+1)$-dimensional chain $\{\lambda_i\}$ such that

$$\partial \lambda_i = K_{\gamma,i} - s(K_{\gamma,i}) - \mu_i ,$$

where $\{\mu_i\}$ is an infinite n-dimensional chain lying in Q_γ. It follows that the sequence $\{K_{\gamma,i} - s(K_{\gamma,i}) - \mu_i\}$ is an infinite n-dimensional cycle lying in the compactum $K_\beta \cup K_\gamma \cap r(M)$ and that this cycle is homologuous to zero in X. Moreover, if we apply the hypothesis that dim $(K_\beta \cap K_\gamma) < n$, we see easily that this cycle is not homologuous to zero in its carrier $K_\beta \cup K_\gamma \cup r(M)$. However the existence of a such infinite cycle implies that the dimension of the space X is greater than n. Thus the proof of the theorem is concluded.

The following **problems** remain open:

1. *Is the theorem true if the notion of ANR-sets in understand in the more general sense, without the hypothesis of compactness?*

2. *Does the theorem remain true if we replace the hypothesis that the uncountable family of sets $\{K_\alpha\}$ consists of ANR-sets, by the weaker hypothesis, that K_α are arbitrary n-dimensional compacta?*

Now I shall present two applications of this theorem: the first to the problem of existence of universal absolute retracts, and the second — to the theory of r-neighbours.

We understand by an universal n-dimensional AR-set every n-dimensional AR-set which topologically contains every other n-dimensional AR-set. Since 1-dimensional AR-sets coincide with the dendrites, that is with locally connected continua which do not contain any simple closed curve, the problem of existence of an 1-dimensional AR-set was solved many years ago by T. WAŻEWSKI ([2]), who constructed a dendrite containing topologically every other dendrite. However the question of existence of n-dimensional universal AR-sets, for $n > 1$, has remained open. By a remark due to K. SIEKLUCKI, our theorem would allow to solve this problem for $n = 2$ in the negative sense, in case we can construct an uncountable family of 2-dimensional AR-sets with the property that none of them topologically contains any 2-dimensional closed subset of another.

I shall give the idea of a construction of such a family. Consider an arbitrary sequence $\{n_k\}$ of natural numbers greater than 1, and let $P_1 = \Delta$ be a triangle in Euclidean 3-space E^3. By T_1 we understand the triangulation of P_1 consisting of the triangle Δ and all its sides and vertices. Consider a system of n_1 triangles $\Delta_1, \ldots, \Delta_{n_1}$ lying in the interior of the triangle Δ and satisfying the following two conditions:

1. The barycenter b_Δ of Δ is the common vertex of $\Delta_1, \ldots, \Delta_{n_1}$.
2. $\Delta_i \cap \Delta_j = (b_\Delta)$ for $i \neq j$.

Now let ε_1 be a positive number and let $\overline{a_\Delta b_\Delta}$ be a segment of length ε_1, perpendicular to the triangle Δ. Consider the system of $3n_1$ triangles $\Delta'_1, \ldots, \Delta'_{3n_1}$ which are spanned by the point a_Δ and by all sides of the triangles $\Delta_1, \ldots, \Delta_{n_1}$. Let us denote by P_2 the polytope

$$R(\Delta, n_1, \varepsilon_1) = \Delta - \bigcup_{i=1}^{n_1} \Delta_i \cap \bigcup_{j=1}^{3n_1} \Delta'_j .$$

Next consider a triangulation T_2 of this polytope and replace each of the triangles T_2 by the polytope $R(\Delta', n_2, \varepsilon_2)$ where ε_2 is a sufficiently small positive number. Thus we obtain a polytope P_3. By iterating this procedure, we obtain a sequence $\{P_k\}$ of 2-dimensional polytopes in E^3 and it is easy to prove that, by a suitable choice of the triangulations $T_1, T_2, \ldots$ and of the numbers $\varepsilon_1, \varepsilon_2, \ldots$, the sequence $\{P_k\}$ converges to a 2-dimensional AR-set, which we denote by $P(\{n_k\})$.

Now let us consider a sequence $\{w_n\}$ of all rational numbers and let us assign to every real number t the increasing sequence $\{n_k(t)\}$ consisting of all the integers n for which $w_n < t$. Setting

$$\Phi(t) = P(\{n_k(t)\}),$$

one obtains a family consisting of $2^{\aleph_0}$ two-dimensional AR-sets with the property that, for $t \neq t'$, none of the 2-dimensional closed subsets of $\Phi(t)$ is topologically included in $\Phi(t')$. By the preceding theorem, we see at once that none of the 2-dimensional ANR-sets could topologically contain all the sets $\Phi(t)$. Consequently a 2-dimensional universal AR-set does not exist.

The other application of our theorem concerns the theory of r-neighbours. (See [1].) We say that a space X is *r-smaller* than a space Y (in symbols: $X \underset{r}{<} Y$) provided X is homeomorphic to a retract of Y, but Y is not homeomorphic to a retract of X. If $X \underset{r}{<} Y$, but no space Z satisfies the condition $X \underset{r}{<} Z \underset{r}{<} Y$, then we say that X is an r-neighbour of Y on the left. It is easy to show that if X is an r-neighbour on the left of the Euclidean 3-cube Q^3, then X must be a 2-dimensional AR-set, which topologically contains all of the sets $\Phi(t)$. However, by our theorem, this is impossible. Consequently the cube Q^3 has no r-neighbours on the left.

Added in proof. The problem 2 is positively solved recently by K. SIEKLUCKI.

References

[1] *K. Borsuk*: Concerning the classification of topological spaces from the stand-point of the theory of retracts. Fundam. Math. *46* (1959), 321–330.

[2] *T. Ważewski*: Sur les courbes de Jordan ne renfermant aucune courbe simple fermée de Jordan. Ann. Soc. Polon. Math. *2* (1923), 49–170.