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ON DIVISION PROBLEMS FOR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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1. Introduction. Let Ω be a domain of the n-space $R^n = \{(x_1, x_2, \ldots, x_n)\}$ and Γ_1, Γ_2 relatively open, bounded portions of the boundary of Ω and such that $\Gamma_1 \supset \bar{\Gamma}_2$.

Let P be a differential polynomial of order m with variables $\partial/\partial x_i$ $(i = 1, 2, \ldots, n)$.

In this note we consider the relation between the following two problems, with respect to Ω, Γ_i $(i = 1, 2)$ and P:

(i) Let Ω_1 be a neighbourhood[1]) of $\bar{\Gamma}_1$ relative to Ω. The problem is then whether there exists a neighbourhood Ω_2 of $\bar{\Gamma}_2$ relative to Ω with the following property of regularity:

(R) For the Cauchy problem,

$$PU = f \text{ on } \Omega, \quad D^\alpha U = g^\alpha, \quad \text{for } |\alpha| \leqq m - 1 \text{ on } \dot{\Omega}_1 \cap \dot{\Omega}$$

the condition $U \in C^\infty(\bar{\Omega}_2)$ follows from the conditions $f \in C^\infty(\bar{\Omega}_1)$ and $g^\alpha \in C^\infty(\dot{\Omega}_1 \cap \dot{\Omega})$.

(ii) What kind of division theorems for the differential operator P' with respect to distributions defined over Ω are valid.

The problem (i) for a strictly convex surface Γ_1 and any P is treated by F. JOHN, B. MALGRANGE [2], and for convex domains by M. ZERNER [3] (also [2]). In particular John and Zerner showed that the problem for a non strictly convex surface includes some complicated results.

On the other hand, the problem (ii) has been considered by several authors since L. SCHWARTZ had proposed such problems in his distribution theory.

Though it seems very interesting to obtain an algebraic characterization of the relations among Ω, Γ_i $(i = 1, 2)$, and P which satisfy the property (R), our purpose in the present note is to establish the equivalence of the affirmative answers of the problems (i) and (ii) with some additional conditions.

2. To interpret the problem (ii), we introduce a topology in the set $C_0^\infty(\Omega)$ of all C^∞-functions with compact supports in Ω.

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[1]) In this note by a neighbourhood of Γ_1 is meant the set $\{x \mid x \in S(y, \varepsilon_y), y \in \Gamma_1\}$, where $S(y, s_y) = \{x \mid |x - y| < \varepsilon_y\}$. Furthermore for any $\varepsilon > 0$ the set $S(\infty, \varepsilon) = \{x \mid |x| > \varepsilon^{-1}\}$ is considered as a neighbourhood of the infinite point.

We consider the following sets of subsets N of $C_0^\infty(\Omega)$ as a base of neighbourhoods of zero:

Let $\{\Omega_i\}$ be a sequence of neighbourhoods of $\overline{\Gamma}_1$ relative to Ω such that

$$\Omega = \Omega_0 \supset \Omega_n \supset \overline{\Omega}_{n+1} \ (n = 1, 2, \ldots)$$

and such that

$$\bigcap_{i=1}^{\infty} \Omega_i = \overline{\Gamma}_1 .$$

Let $\{\varepsilon_i(x)\}$ be a sequence of positive continuous functions in Ω and $\{\alpha_i\}$ an increasing sequence of positive integers.

Now let $N = N(\{\Omega_i\}, \{\varepsilon_i(x)\}, \{\alpha_i\})$ be the set

$$\{f \mid f \in C_0^\infty(\Omega), \sum_{|\alpha| \leqq \alpha_i} |D^\alpha f(x)| \leqq \varepsilon_i(x) \text{ for } x \in \Omega_i - \overline{\Omega}_{i+1}, (i = 0, 1, 2, \ldots)\} .$$

Denote the topological vector space thus obtained by $D(\Omega, \overline{\Gamma}_1)$ and its dual topological vector space by $D'(\Omega, \overline{\Gamma}_1)$. Then we see that $D'(\Omega, \overline{\Gamma}_1)$ is a Montel space which consits of all distributions $T \in D'(\Omega)$ such that the local order of T, i. e., the order $\varphi\ T(\varphi \in C_0^\infty(\Omega))$ increases without any restriction as the support of φ approaches $\overline{\Gamma}_1$, but does not do so when the support of φ approaches $\dot{\Omega} - \overline{\Gamma}_1$. Therefore $D'(\Omega, \Gamma) = D'(\Omega)$, if $\Gamma = \dot{\Omega}$, and if $\Gamma = \emptyset$ then $D'(\Omega, \Gamma) = D'_F(\Omega)$, the space of all distributions of finite order.

Now one of our problem can be stated as follows: What kind of relations among Ω, Γ_1, Γ_2 and P imply that

$$P'\ D'(\Omega, \Gamma_1) \supset D'(\Omega, \Gamma_2) .$$

I considered such problems in order to understand the feature of the problem with respect to the regularity of solution of the Cauchy problem or with respect to the prolongation of regularity of solutions of differential equation.

3. We shall prove that the answer of our division problem is in the affirmative whenever the whole boundary of Ω is non-characteristic and the condition (R) is satisfied for Ω, Γ_i $(i = 1, 2)$ and P.

To show this, let us assume these conditions. Then we have the following lemma:

Lemma. *For any neighbourhood Ω_1 of Γ_1, there exists a neighbourhood Ω_2 of $\overline{\Gamma}_2$ such that if $F \in \mathscr{E}'(\Omega_2)$, a solution $S : P'S = F$ has the following property: for some m'*

$$S \in D'(\Omega) \cap (C^{m'})'\ (\overline{\Omega - \overline{\Omega}_1}) .$$

Proof. Let B be the Fréchet space

$$\{f \mid f \in C^m(\overline{\Omega}),\ Pf \in C^\infty(\overline{\Omega}_1),\ D^\alpha f \in C^\infty(\dot{\Omega} \cap \dot{\Omega}_1) \text{ for } |\alpha| \leqq m - 1\} .$$

Then by our hypothesis we see that for some neighbourhood Ω_2 of $\overline{\Gamma}_2$ we have $f \in C^\infty(\overline{\Omega}_2)$, if $f \in B$.

Since the mapping from B into $C^\infty(\overline{\Omega}_2), f \to f \mid \overline{\Omega}_2$, is closed, it is also continuous by the closed graph theorem in Fréchet spaces.

Hence for any compact set K of Ω_2 and any integer k there exist integers p and q such that for some C_1 and for any $f \in B$

$$\|f\|_{C^m(\bar{\Omega})} + \|Pf\|_{C^p(\bar{\Omega}_1)} + \sum_{|\alpha| \leqq m-1} \|D^\alpha f\|_{C^q(\dot{\Omega}_1 \cap \dot{\Omega}_2)} \geqq C_1 \|f\|_{C^k(K)}.$$

In particular, by Malgrange's theorem, if $f \in C_0^\infty(\Omega)$ then for any ε and for some m' and $C_2 > 0$,

$$\|Pf \exp(\varepsilon|x|)\|_{L_2{}^{m'}(\bar{\Omega})} + \|Pf\|_{C^p(\bar{\Omega}_1)} \geqq C_2 \|f\|_{C^k(K)}.$$

Therefore for any $F \in \mathscr{E}'(K)$ there exists a solution S such that

$$P'S = F \text{ on } \Omega, \quad S \in D'(\Omega) \cap (C_0^{m'})' (\overline{\Omega - \bar{\Omega}_1}).$$

Here we remark that the number m' is independent of Ω_1.

Using this lemma and Malgrange's consideration (the method of Mittag-Leffler) we show that for any $\Gamma : \bar{\Gamma} \subset \Gamma_2$, $P' D'(\Omega, \Gamma_1) \supset D'(\Omega, \Gamma)$.

A $T \in D'(\Omega, \bar{\Gamma})$ may then be decomposed into

$$T = T_0 + T_1 + T_2 + \dots + T_n + \dots,$$

where $T_0 \in D'_F(\Omega)$, $T_i \in \mathscr{E}'(\Omega''_i - \bar{\Omega}''_{i+1})$. Here Ω''_i is a neighbourhood of $\bar{\Gamma}_2$ such that $\cap \Omega''_i = \bar{\Gamma}_2$.

On the other hand, there exists a sequence of bounded domains Ω'_i such that $\cup \Omega'_i = \Omega$, $\bar{\Omega}'_i \subset \Omega'_{i+1}$ and such that for any distributions S with

$$P'S = 0 \quad \text{on} \quad \Omega'_{i+1}, \quad S \in (C^{m'})' (\bar{\Omega}'_{i+1})$$

and for any $\varepsilon > 0$, there is a distribution S' with the following properties:

$$P'S' = 0 \quad \text{on} \quad \Omega'_{i+2}, \quad S' \in (C^{m''})' (\bar{\Omega}'_{i+2}),$$
$$\|S - S'\|_{(C^{m''})'(\bar{\Omega}_i')} \leqq \varepsilon$$

for some $m'' \geqq m'$.

Furthermore, we may assume that $\bar{\Omega}_i \cap \bar{\Omega}'_{i+1} = \emptyset$, for some sequence of neighbourhoods Ω_i of Γ_1, and that Ω_i and Ω''_i possess the property described in (R).

Then we can find S_i $(i = 0, 1, 2, \dots)$ such that

$$P'S_0 = T_0, \quad S_0 \in D'_F(\Omega),$$
$$P'S_i = T_i \quad (i = 1, 2, \dots), \quad S_i \in D'(\Omega) \cap (C^{m'})' (\overline{\Omega - \bar{\Omega}_i}).$$

Furthermore, since $P'S_i = 0$ on Ω'_{i+1}, there exists an S'_i such that

$$P'S'_i = 0 \quad \text{on} \quad \Omega'_{i+2}, \quad \|S_i - S'_i\| (C^{m''})' (\bar{\Omega}'_i) \leq 1/i^2, \quad S'_i \in (C^{m''})' (\bar{\Omega}'_{i+1}).$$

Taking $\varphi_i \in C_0^\infty(\Omega)$ such that $\varphi_i(\mu) \equiv 1$ for $\mu \in \Omega'_i$ and $\equiv 0$ for $\mu \in \Omega'_{i+2}$, we define

$$\bar{S} = S_0 + \sum_{i=1}^{\infty} (S_i - \varphi_i S'_i).$$

Then $\bar{S} \in D'(\Omega, \bar{\Gamma}_1)$. For in the domain $\Omega - \bar{\Omega}_i$ and for $j \geqq i$, we have $S'_j \in (C^{m''})'$. $. (\Omega'_i)$, and therefore

$$\sum_{j=1}^{\infty}(S_j - \varphi_j S'_j) = \sum_{j=1}^{i-1}(S_j - \varphi_j S'_j) + \sum_{j=i}^{\infty}(S_j - \varphi_j S'_j) \in (C^l)'\,(\overline{\Omega - \bar{\Omega}_i}) + + (C^{m''})'\,(\overline{\Omega - \bar{\Omega}_i}) \quad \text{for some} \quad l > 0\,.$$

Furthermore $P'\bar{S} = \sum P'(\varphi_i S'_i) \in (C_0^{m''})'\,(\Omega)$, so that we can find a distribution $\bar{\bar{S}} \in$ $\in D'_F(\Omega)$ such that $P'\bar{\bar{S}} = \sum_i P'(\varphi_i S'_i)$. Thus $\dot{S} = \bar{S} - \bar{\bar{S}}$ is the desired distribution.

4. We shall prove the converse assertion of the result established in 3. For the sake of simplicity of description we assume that there exists a non-tangential vector ξ such that $x + \varepsilon(x)\xi \subset \Omega$ for any small $\varepsilon(x)$ and furthemore that $\dot{\Omega} \cap \dot{\Omega}_1$ is in a hyperplane which is non characteristic and that Ω is convex.

Suppose, contradicting the condition (R), that there exists a neighbourhood Ω'_1 of Γ_1 ($\Omega_1 \subset \Omega'_1$) such that for any neighbourhood Ω_2 of $\bar{\Gamma}_2$, there is a function f with the following properties:

$$f \in C^m(\bar{\Omega})\,,$$

$$D^\alpha f \in C^\infty(\dot{\Omega}_1 \cap \dot{\Omega}) \quad \text{for} \quad |\alpha| \leqq m - 1 \quad \text{and} \quad Pf \in C^\infty(\bar{\Omega}'_1)\,, \quad \text{but} \quad f \notin C^\infty(\bar{\Omega}_2)\,.$$

Let $\{\Omega_{2,1}\}$ be a sequence of neighbourhoods of $\bar{\Gamma}_2$ such that

$$\Omega_{2,i} \subset \{x \mid \text{dist}\,(x, \Gamma_2) < 1/i\}\ ;$$

denote the f corresponding to $\Omega_{2,i}$ by f_i. Since $\dot{\Omega} \cap \dot{\Omega}'_1$ is a C^∞-surface, there exists a $\psi \in C^\infty\,(U\,(\dot{\Omega} \cap \dot{\Omega}'_1))$ such that $\{x \mid \psi(x) = 0\}$ contains $\dot{\Omega} \cap \dot{\Omega}'_1$, and grad $\psi \neq 0$, where $U(\dot{\Omega} \cap \dot{\Omega}'_1)$ is a neighbourhood of $\dot{\Omega} \cap \dot{\Omega}'_1$ in R^n.

Furthermore, since $\dot{\Omega} \cap \dot{\Omega}'_1$ is non-characteristic, for the solution $Pf = g \in C^\infty(\bar{\Omega})$, $D^\alpha f = h^\alpha \in C^\infty(\dot{\Omega}'_1 \cap \dot{\Omega})$ we have that the values

$$\left(\sum \frac{\partial\psi}{\partial x_i}\frac{\partial}{\partial x_i}\right)^\beta f \quad \text{on} \quad \dot{\Omega} \cap \dot{\Omega}'_1$$

are determined by g, h^α; denote these by $f^{(\beta)}$. Finally assuming Ω is in $\{x \mid \psi(x) > 0\}$, put

$$\Omega^- = \{x \mid 0 \geqq \psi(x) \geqq -\,\delta\}\,.$$

Then we can extend the function f to an $\bar{f}$ such that

$$\bar{f} \in C^\infty(\bar{\Omega}^-)\,, \quad P\bar{f} \in C^\infty(\Omega_1 \cup \Omega^-)\,,$$

since we can apply the well known theorem concerning Fréchet space ([4]) to the mapping from

$$C^\infty(\bar{\Omega}^-) \quad \text{into} \quad \Pi\,C^\infty(\dot{\Omega} \cap \dot{\Omega}_1) : f \to \{f^{(\beta)}\} \quad (\text{weak topology})\,.$$

Setting $\bar{\bar{f}}_i(x) = \bar{f}_i(x - i^{-1}\xi)\,\psi_i(x)$ for some $\psi_i(x) \in C_0^\infty(\Omega)$, we may assume that

$$f_i \in C_0^m(\Omega)\,, \quad Pf_i \in C^\infty(\Omega'_1)$$

but

$$f_i \notin C^\infty(\Omega_{2,i})\,.$$

Furthermore by using a convolution operator with sufficiently small support, we may assume that

$$f_i \in C_0^i(\Omega - \bar{\Omega}_{2,i}), \quad \text{but} \quad f_i \notin H^{i+n}(\Omega_{2,i} - \bar{\Omega}_{2,i+1}).$$

Now we can show that there exist distributions F_i of order $\leqq s_i = i + n$ such that

$$\text{supp} \quad F_i \subset \Omega_{2,i} - \bar{\Omega}_{2,i+1}, \quad F_i(f_i * \varphi_\varepsilon) \to \infty \quad \text{as} \quad \varepsilon \to 0,$$

where φ_ε is a C^∞-function such that $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\varepsilon^{-1}x)$, $\varphi(x) \geqq 0$, $\text{supp}\,\varphi \subset \subset \{x \mid (x) \leqq |\}$ and $\int \varphi \, dx = 1$. For, if for any $F \in (H^{i+n})'(\Omega_{2,i} - \bar{\Omega}_{2,i+1})$, $|F(f_i * * \varphi_\varepsilon)| < k$ for some constant k which depends on F, but is independent of ε, then $f_i * \varphi_\varepsilon$ converges weakly to some element of $H^{i+n}(\Omega_{2,i} - \bar{\Omega}_{2,i+1})$. But $f_i * \varphi_\varepsilon$ converges to f_i in $C_0^i(\Omega)$, therefore f_i itself belongs to $H^{i+n}(\Omega_{2,i} - \bar{\Omega}_{2,i+1})$, which implies a contradiction.

Let $z_1 = s_1$, $z_2 = s_{z_1}$, ..., $z_i = s_{z_{i-1}}$, and let $F = F_1 + F_{z_1} + \ldots + F_{z_i} + + \ldots$ Then obviously $F \in D'(\Omega, \Gamma_2)$. Assume that there exists a distribution $S \in \in D'(\Omega, \bar{\Gamma}_1)$ such that $P'S = F$. Then $S(P\varphi) = F(\varphi)$ for any $\varphi \in D(\Omega)$. Therefore there exists a neighbourhood N of O in $D'(\Omega, \Gamma_1)$ such that if $P\varphi \in N$, then $|F(\varphi)| \leqq 1$. But since $Pf_i \in C_0^{i-m}(\Omega) \cap C^\infty(\Omega_1')$, whenever $i > k$ and $\varepsilon < \varepsilon_i$, for some k, δ_i and ε_i

$$P(\delta_i f_i * \varphi_\varepsilon) = \delta_i(Pf_i) * \varphi_\varepsilon \subset N,$$

and hence $|F(\delta_i f_i * \varphi_\varepsilon)| < 1$ for $i > k$ and for $\varepsilon < \varepsilon_i$. Furthermore, since the order of $F_{z_i} \leqq z_j$ $(i < j)$ and $f_{z_l} \in C_0^{z_l}(\Omega - \bar{\Omega}_{2,z_l})$ for $z_l > k$, therefore

$$|(F_1 + F_{z_1} + \ldots + F_{z_{l-1}})(\delta_{z_l} f_{z_l} * \varphi_\varepsilon)| \leqq K \quad \text{as} \quad \varepsilon \to 0 \quad \text{for some } K$$

and

$$|F_{z_l}(\delta_{z_l} f_{z_l} * \varphi_\varepsilon)| \to \infty \quad \text{as} \quad \varepsilon \to 0$$

therefore for some $\varepsilon < \varepsilon_i$

$$|F(\delta_{z_l} f_{z_l} * \varphi_\varepsilon)| \geqq (2 + K) - K = 2$$

which is a contradiction. Thus we obtain that $P'D'(\Omega, \Gamma_1) \not\supset D'(\Omega, \Gamma)$.

5. In the previous sections a connection between the existence of solution of differential operators and the regularity of solution of Cauchy problems was considered. In this section a relation between the former and the regularity of solution of Cauchy problems in a more general sense is noted.

From now on we assume that Γ is a closed $(\Gamma = \bar{\Gamma})$ and connected component of the boundary of Ω. Then we have the following proposition: $P'D'(\Omega, \Gamma) = D'(\Omega, \Gamma)$ if and only if

1. Ω is P-convex, i. e., for any compact subset K of Ω there exists a compact subset K' such that if supp $PT \subset K$ and $T \in \mathscr{E}'(\Omega)$, then supp $T \subset K'$.

2. For any neighbourhood Ω_1 of Γ, there exists another neighbourhood Ω_2 of $\bar{\Gamma}$ such that if $\varphi \in \mathscr{E}'(\Omega)$ satisfies $P\varphi \in C^\infty(\Omega_1)$, then $\varphi \in C^\infty(\Omega_2)$.

Here we assume that if Γ contains the point at infinity then our neighbourhood of Γ contains $\{x \mid x \in \Omega, |x| > L\}$ for some positive L.

The proof of this proposition is analogous to that of the previous. The difference is in the part corresponding to 3.

Let Ω_1 and Ω_2 be chosen such that they satisfy condition 2 and let Ω_0 and Ω_3 be other neighbourhoods of Γ such that $\overline{\Omega}_3 \subset \Omega_2$, $\overline{\Omega}_1 \subset \Omega_0$.

Here obviously we may suppose that $\overline{\Omega}_2 \subset \Omega_1$. By our assumption even if Γ contains the point at infinity, $\Omega_0 - \overline{\Omega}_3$ is bounded.

Now consider the mapping from the space B_1 to $C^\infty(\Omega_2 - \overline{\Omega}_3)$, where $B_1 = \{f \mid f \in C^\infty$ (a fixed small neighbourhood $U(\dot{\Omega}_3 - \Gamma)$ of $\dot{\Omega}_3 - \Gamma$) and $Pf \in C^\infty(\Omega - \overline{\Omega}_3)\}$.

This mapping is defined and continuous.

For, let φ be a function in $C_0^\infty(\Omega_0)$ such that

$$\varphi(x) \equiv 1 \quad \text{on} \quad \Omega_0 - \overline{\Omega}_3 \cup U(\dot{\Omega}_3 - \Gamma) \cup U(\dot{\Omega}_0 - \Gamma),$$
$$\varphi(x) \equiv 0 \quad \text{on} \quad \Omega_3 \cup U_1(\dot{\Omega}_3 - \Gamma) \cup U_1(\dot{\Omega}_0 - \Gamma),$$

where $U_1(\dot{\Omega}_3 - \Gamma)$ is a neighbourhood of $\dot{\Omega}_3 - \Gamma$ smaller than $U(\dot{\Omega}_3 - \Gamma)$.

Then for any $f \in B_1$

$$\varphi f \in \mathscr{E}'(\Omega) \quad \text{and} \quad P\varphi f \in C^\infty(\Omega_1)\,;$$

therefore $\varphi f \in C^\infty(\Omega_2)$ by our assumption, and in particular

$$f \in C^\infty(\Omega_2 - \overline{\Omega_3 \cup U_1(\dot{\Omega}_3 - \Gamma)}).$$

Since B_1 and $C^\infty(\Omega_2 - \overline{\Omega}_3)$ are Fréchet spaces, by the method used in 3, for any $T \in \mathscr{E}'(\Omega_2 - \overline{\Omega}_3)$, we obtain a distribution S' such that

$$P'S' = T \quad \text{in} \quad \Omega_0 - \overline{\Omega}_3\,, \quad S' \in (H^{m'})'\,(\Omega_0 - \overline{\Omega}_1) \cap D'(\Omega_0 - \overline{\Omega}_3)\,.$$

Then setting $S = \varphi S'$, $P'S = T + T_1 + T_2$, where $T_1 \in D'_F(U(\dot{\Omega}_0 - \Gamma))$ and $T_2 \in D'(U(\dot{\Omega}_3 - \Gamma))$. Using this S, we can complete our proof as in 3.

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