MAPPINGS OF PROXIMITY STRUCTURES

C. H. DOWKER

London

We consider proximity structures without the usual requirement of symmetry. Given a function $f : X \to Y$, there are three mappings f^q, f^0 and f^c of the proximity structures of X to those of Y related, respectively, to the notions of continuity, openness and closedness. The mappings f^0 and f^c do not in general preserve symmetry.

We also solve a problem of YU. M. SMIRNOV ([1], page 546) by giving an example of a symmetric proximity space which does not have a finest symmetric uniform structure inducing its proximity structure. The example is the product of two infinite spaces, with the product proximity structure.

A proximity structure in a set X is a relation $<$ in the set of all subsets of X, satisfying the following axioms:

1. $A < B$ implies $A \subset B$.
2. $A \subset B < C \subset D$ implies $A < D$.
3. $A_i < B$ for all $i \in I$, I finite, implies $\bigcup_i A_i < B$; $A < B_i$ for all $i \in I$, I finite implies $A < \bigcap_i B_i$.
4. $A < C$ implies that there exists B such that $A < B < C$.

Taking I void in axiom 3, we see that $\emptyset < A < X$ for every set A in X. A proximity structure $<_1$ is called finer than $<$ if $A < B$ implies $A <_1 B$. A set X has a finest proximity structure: the discrete structure in which $A < B$ whenever $A \subset B$. It also has a least fine proximity structure in which $A < B$ only if $A = \emptyset$ or $B = X$. A set X, together with a proximity structure $<$ in it, is called a proximity space.

The proximity structure $<'$, such that $A <' B$ if and only if $X \setminus B < X \setminus A$, is called the conjugate of $<$. The proximity structure $<$ is called symmetric if $<' =$ $= <$. We shall not assume an axiom of symmetry.

A proximity structure in X induces a topology in X, a set A being a neighbourhood of a point x if $(x) < A$. A finer proximity structure induces a finer topology.

If $f : X \to Y$ is a function and $<$ is a proximity structure in Y, let $A <_1 B$, for A and B in X, if there is some set C in Y such that $f(A) < C$ and $f^{-1}C \subset B$. Then $<_1$ is a proximity structure in X, called $f^{-1}(<)$. If $<$ is symmetric, so is $f^{-1}(<)$. If a given proximity structure $<_0$ in X is finer than $f^{-1}(<)$, f is called a proximally continuous function from $(X, <_0)$ to $(Y, <)$. The inverse image of the topology T induced by $<$

is the topology induced by $f^{-1}(<)$. Hence a proximally continuous function is continuous.

A uniform structure in a set X is a family $V = \{u\}$ of functions from X to the set 2^X of all subsets of X, satisfying the following axioms:

1. For each $x \in X$ and each $u \in V$, $x \in u(x)$.
2. If $u \in V$ and $u < v$ (i. e., $u(x) \subset v(x)$ for all x), then $v \in V$.
3. If $u_i \in V$ for $i \in I$, I finite, then $\bigcap_i u_i \in V$.
4. Given $u \in V$ there exists $v \in V$ such that $v^2 < u$, i. e., if $y \in v(x)$, $v(y) \subset u(x)$.

In axiom 3, $\bigcap u_i$ is the function which assigns to the point x the set $\cap\, u_i(x)$. The case of axiom 3 when I is void states that the maximal function $\mathit{1}$, defined by $\mathit{1}(x) = X$ for all $x \in X$, belongs to V. Thus V is not empty. A uniform structure W is called finer than V if $V \subset W$. There is a finest uniform structure in X consisting of all functions satisfying axiom 1, and there is a least fine uniform structure consisting only of the function $\mathit{1}$.

The function v', defined by $v'(x) = \{y : y \in X,\ x \in v(y)\}$, is called the conjugate of v. The family $V' = \{u'\}$ of conjugates of functions u in the uniform structure V is itself a uniform structure, called the conjugate of V. The uniform structure V is called symmetric if $V' = V$.

A uniform structure V induces a proximity structure $<$ as follows: $A < B$ if there exists $u \in V$ such that $\bigcup_{x \in A} u(x) \subset B$.

If $f : X \to Y$ is a function and V is a uniform structure in Y, there is a uniform structure $f^{-1}V$ in X defined as follows: Let $u \in f^{-1}V$ if there exists $v \in V$ such that for each $x \in X$, $u(x) \supset f^{-1}vf(x)$. If a given uniform structure U in X is finer than $f^{-1}V$, f is called a uniformly continuous function from (X, U) to (Y, V). If V induces the proximity structure $<$, then $f^{-1}V$ induces $f^{-1}(<)$. Hence a uniformly continuous function is proximally continuous.

Given a function $f : X \to Y$ and given a proximity structure $<$ in X, we define the quotient proximity structure $f^q(<)$ to be the finest proximity structure $<_1$ in Y for which $f : (X, <) \to (Y, <_1)$ is proximally continuous. Similarly a quotient topology $f^q(T)$ and a quotient uniform structure $f^q(V)$ can be defined. If $<$ is induced by a uniform structure V in X, $f^q(<)$ is induced by $f^q(V)$. If T is the topology induced by $<$ in X, $f^q(T)$ is finer, and in some cases strictly finer, than the topology induced by $f^q(<)$. If $<$ or V is symmetric, so is $f^q(<)$, respectively $f^q(V)$.

Given a function $f : X \to Y$ and given a proximity structure $<$ in X, we define the open image $f^0(<)$ to be the least fine proximity structure $<_1$ in Y such that $f(A) <_1 f(B)$ whenever $A < B$. The open image of a topology or of a uniform structure is similarly defined. If Y has a given proximity structure $<_0$, f is called proximally open if $<_0$ is finer than $f^0(<)$. Open functions and uniformly open functions are similarly defined. If $<$ induces the topology T, $f^0(<)$ induces $f^0(T)$. If $<$ is induced by a uniform structure V, the proximity structure induced by $f^0(V)$ is finer, and may be strictly finer, than $f^0(<)$. Thus a proximally open function is open, and a uniformly open function is proximally open.

If $<'$ is the conjugate of $<$, $f^0(<')$ is not necessarily the conjugate of $f^0(<)$. In particular, if $<$ is symmetric, $f^0(<)$ need not be symmetric. Similarly, the symmetry of the uniform structure V does not imply symmetry of $f^0(V)$.

Given a function $f: X \to Y$ and given a proximity structure $<$ in X, we define the closed image $f^c(<)$ to be the least fine proximity structure $<_1$ in Y such that $A <_1 Y \setminus f(B)$ whenever $f^{-1}A < X \setminus B$. The closed image of a topology or of a uniform structure is similarly defined. If Y has a given proximity structure $<_0$, f is called proximally closed if $<_0$ is finer than $f^c(<)$. Closed functions and uniformly closed functions are similarly defined. Every uniformly closed function is proximally closed, but it need not be closed.The closed image of a symmetric proximity structure or uniform structure need not be symmetric.

If $u: X \to 2^X$ is any function, we say that a set $A \subset X$ is u-small if, for each pair of points x_1 and x_2 in A, $x_2 \in u(x_1)$. A uniform space (X, V) is called totally bounded if for each $u \in V$ there exists a finite decomposition of X into u-small sets.

If $\{X_\omega, <_\omega\}_{\omega \in \Omega}$ is any family of proximity spaces, the product proximity structure $\Pi <_\omega$ is defined to be the least fine proximity structure in ΠX_ω such that each projection $\pi_\omega : \Pi X_\omega \to X_\omega$ is proximally continuous. The product topology and product uniform structure are similarly defined. If $<_\omega$ induces the topology T_ω, then $\Pi <_\omega$ induces ΠT_ω. If V_ω induces $<_\omega$ and if all but one of the uniform spaces (X, V_ω) are totally bounded, then ΠV_ω induces $\Pi <_\omega$. The hypothesis of total boundedness can not be omitted.

For example, let Z be the space of integers, let U be the uniform structure of finite decompositions of Z and let V be the finest uniform structure of Z. Then U and V induce the same discrete proximity structure $<$ in Z. Since (Z, U) is totally bounded, the symmetric uniform structures $U \times U$, $U \times V$ and $V \times U$ induce the same proximity structure $< \times <$ in $Z \times Z$. But this proximity structure is strictly less fine than the discrete proximity structure induced by $V \times V$. Since $V \times V$ is the only uniform structure in $Z \times Z$ which is finer than both $U \times V$ and $V \times U$, therefore there is no finest symmetric uniform structure inducing the proximity structure $< \times <$ in $Z \times Z$.

Reference

[1] *Ю. М. Смирнов:* О пространствах близости. Мат. сб., *31* (1952), 543—574.