Beloslav Riečan

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ON MEASURABLE SETS IN TOPOLOGICAL SPACES

B. RIEČAN

Bratislava

In measure theory the following theorem is well-known: If X is a metric space and μ is a Carathéodory outer measure (i.e., $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $\operatorname{dist}(A, B) > 0$), then every open set is μ-measurable. In this report we present several similar theorems in topological spaces.

All outer measures will be defined on the system of all subsets of a space X. A set $A \subset X$ is called μ-measurable (where μ is an outer measure) iff $\mu(E) = \mu(E \cap \cap A) + \mu(E - A)$ for any $E \subset X$.

The first theorem is formulated for an abstract space. From it all the other theorems easily follow.

Theorem 1. *Let X be a non empty set, $\mathbf{R}$ be a symmetric relation defined on the system of all subsets of X with the following property: If $E\mathbf{R}F$, $E_1 \subset E$, $F_1 \subset F$, then $E_1\mathbf{R}F_1$. Let μ be an outer measure such that $\mu(E \cup F) = \mu(E) + \mu(F)$ whenever $E\mathbf{R}F$. Let $C = \bigcap_{n=1}^{\infty} V_n$, $V_{n+1} \subset V_n$, $C\mathbf{R}(X - V_n)$, $(V_n - V_{n+1})\,\mathbf{R}V_{n+2}$ $(n = 1, 2, \ldots)$. Then the set C is μ-measurable.*

Theorem 2. *Let X be a regular topological space. Let μ be an outer measure such that $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever there are open disjoint sets U, V with $\bar{A} \subset U$, $\bar{B} \subset V$ ($\bar{A}$ is the closure of A). Then every compact G_δ set is μ-measurable.*

Theorem 2 can be obtained from Theorem 1 by introducing the following relation: $E\mathbf{R}F$ iff there are open disjoint sets U, V such that $\bar{E} \subset U$, $\bar{F} \subset V$.

Theorem 3. *Let X be a locally compact Hausdorff topological space. Let μ be an outer measure such that $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever A, B are bounded sets with disjoint closures. Then every compact G_δ set is μ-measurable.*

In this case it is sufficient to define the relation $\mathbf{R}$ as follows: $E\mathbf{R}F$ iff E, F are bounded sets with disjoint closures.

Theorem 4. *Let X be a uniform space with the uniformity $\mathscr{U}$. Let μ be an outer measure for which $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever there is a $V \in \mathscr{U}$ such that $A \times B \subset X \times X - V$. Then every compact G_δ set is μ-measurable.*

To obtain Theorem 4 from Theorem 1 put $E\mathbf{R}F$ iff there is a $V \in \mathcal{U}$ such that $A \times B \subset X \times X - V$.

Theorem 5. *Let X be a normal topological space. Let μ be an outer measure for which $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $\bar{A} \cap \bar{B} = \emptyset$. Then every closed G_δ set is μ-measurable.*

Here $E\mathbf{R}F$ iff $\bar{E} \cap \bar{F} = \emptyset$.

Notice that some results of W. W. Bledsoe, A. P. Morse, N. Bourbaki and Z. Riečanová published in papers [1], [2] and [3] follow from our Theorem 1. Theorem 5 is known and can be generalized to φ-normal spaces ([1]). Theorem 2 is valid even when the assumption of regularity of X is replaced by the weaker assumption of μ-regularity of X. A topological space is μ-regular iff for any open set U, any compact set $C \subset U$, any set E of finite μ-measure and any $\varepsilon > 0$ there are an open set V and a closed set D such that $D \subset C$, $D \subset V$, $\bar{V} \subset U$, $\mu(E \cap (C - D)) < \varepsilon$.

A detailed elucidation of our results including proofs will appear in the journal Časopis pro pěstování matematiky.

References

[1] *W. W. Bledsoe and A. P. Morse:* A topological measure construction. Pacif. J. Math. *13* (1963), 1067—1084.

[2] *N. Bourbaki:* Sur un théorème de Carathéodory et la mesure dans les espaces topologiques. C. R. Acad. Sci. Paris *201* (1935), 1309—1311.

[3] *Z. Riečanová:* О внешней мере Каратэодори. Mat. fyz. časopis SAV *12* (1962), 246—252.