CONCERNING A CATEGORIAL APPROACH TO TOPOLOGICAL AND ALGEBRAIC THEORIES

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Praha

The present note can be considered as a survey of some results concerning relations of algebraic and topological theories. The method used will be categorial since we may often associate with an intuitive notion of a theory an exact notion of a concrete category; e.g. with the theory of topological spaces we associate the category of topological spaces and continuous mappings, with the theory of groups the category of groups etc. One kind of relations between theories can be described by means of full embeddings. If a category $\mathfrak{K}$ can be fully embedded in $\mathfrak{L}$, then it corresponds to an intuitive meaning that the theory associated with $\mathfrak{L}$ is more general then the one associated with $\mathfrak{K}$. We shall consider two – in some sense extreme – kinds of embeddings.

(1) full embedding – we shall call it a representation,

(2) full embedding preserving underlying sets and the actual form of the mappings – we call it a realization.

Representations. In [5] J. R. Isbell proved that every category of algebras can be represented in a category of algebras with unary operations only. In [2] it is shown that every category of algebras can be represented e.g. in the category of algebras with two unary operations (denoted by $\mathfrak{A}(1, 1)$). Following [2] and an Isbell's definition ([6]) we shall call a category boundable if it is representable in $\mathfrak{A}(1, 1)$. Specifying set theory, it has been proved in [5] that the category of compact Hausdorff spaces is boundable. In [3] it has been proved that the category of topological spaces and their continuous mappings and many other categories appearing in topology are boundable. In [9] there is constructed a concrete category U such that every concrete category is representable in U. Thus, the problem raised in [5], wheter there exists a concrete category which is not boundable, has been reduced to the question, whether the category U is boundable or not. Nevertheless, this problem remains open. Trying to find a category which is not boundable, it has been shown that various categories are boundable ([2], [3]).

From the above results it follows that the categories in which $\mathfrak{A}(1,1)$ is representable – call them binding – are sufficiently rich. It follows from [1] that the category of topological spaces with local homeomorphisms is binding. The paper [8] is in a close relation to this problem, as it shows categories in which every one object category can be represented, which are thus natural candidates for binding categories. The negative results of [8] show that some topologically defined categories are not binding.

Realizations. Consider a countable set A and choose a point $a \in A$. The semigroup of all mappings of A into itself is evidently isomorphic with the semigroup of such $f : A \to A$ for which $f(a) = a$ and $f(A \setminus (a)) \subset A \setminus (a)$. Hence, this two systems of mappings are not distinguishable by their algebraic structure, while their intrinsic structures differ substantially (e.g. every mapping of the second system has a fixed point). In our terminology, we may say that they do not differ from the point of view of representation, while they differ from the point of view of realization. It is possible to show ([9]) that there is a concrete category U such that every concrete category is realizable in U. However, no category which is rich enough to realize all the generally discussed categories may be defined by a simple structure. We shall show that certain families of categories can be realized in a substantially simpler defined ones.

Paragraph 1 contains examples and definitions. In paragraph 2 we deal with representation of categories and in paragraph 3 with their realization.

1. Examples and definitions

To describe the intuitive notion of a "topological theory" in a simple way, we shall introduce categories defined by functors. First, we begin with examples.

1. *Topological spaces.* A topology on a set X is a family of subsets of X (satisfying some axioms). We may describe it as follows: we have a subset $t \subset P(X)$, where $P(X)$ denotes the power set of X. If (X, t) and (X', t') are topological spaces, a mapping $f : X \to X'$ is said to be continuous if and only if the preimage of every set in t' is in t. It suggests to define a contravariant functor P^- from the category of sets into itself, associating with every X its power set $P(X) = P^-(X)$ and with every $f : X \to Y$ the mapping $P^-(f) : P^-(Y) \to P^-(X)$ defined by $P^-(f)(A) = f^{-1}(A)$. Now, we may define the continuous mappings $f : (X, t) \to (X', t')$ as those which satisfy the condition $P^-(f)(t') \subset t$.

2. *Proximity spaces.* A proximity on a set X is a binary relation r on $P(X)$ "to be near". If we define a functor P^+, associating with every set X its power set $P^+(X) = P(X)$ and with every $f : X \to Y$ the mapping $P^+(f) : P^+(X) \to P^+(Y)$ defined by $P^+(f)(A) = f(A)$ for every $A \subset X$, we may describe the proximity mappings from (X, r) into (X', r') as those $f : X \to X'$ for which $P^+(f)$ preserves the relations (i.e. such that $(P^+(f)(A), P^+(f)(B)) \in r'$ whenever $(A, B) \in r$). We remark that the subset $t \subset P^-(X)$ in the example 1) may be considered as a unary relation on $P^-(X)$; then continuous mappings are those f for which $P^-(f)$ preserve the unary relations.

3. *Uniform spaces.* A uniformity on X is a subset (unary relation) s of $P(X \times X)$. Denote by Q a functor associating with every X its square $X \times X$ and with every $f : X \to Y$ the mapping $Q(f)$ defined by $Q(f)(x, y) = (f(x), f(y))$. We see easily that $f : (X, u) \to (X', u')$ is uniformly continuous if and only if $P^- \circ Q(f)(u') \subset u$; i.e. if $P^- \circ Q(f)$ preserves the unary relation.

4. *Relational systems and algebras.* An A-nary relation on a set X is a subset of X^A. A type $\Delta = \{\alpha_\beta \mid \beta < \gamma\}$ is a set of ordinals indexed by all ordinals less than a given one. A relational system r of a type Δ on a set X is a system $r = \{r_\beta\}$, where r_β is a α_β-nary relation on X, i.e. $r_\beta \subset X^{\alpha_\beta}$. Let r and r' be relational systems of the same type on X and X' respectively. A mapping $f : X \to X'$ is said to be compatible (more exactly rr'-compatible), if for every $\beta < \gamma$ and every $\{x_i\} \in r_\beta$, $\{f(x_i)\} \in r'_\beta$.

In this sense algebraic structure of a type $\Delta = \{\alpha_\beta \mid \beta < \gamma\}$ may be considered as a special type of a relational system of the type $\bar{\Delta} = \{\alpha_\beta + 1 \mid \beta < \gamma\}$. The homomorphisms, then, are exactly the compatible mappings.

5. *Topological groups.* A structure of a topological group on a set X consists of a topology t on X and a binary operation (ternary relation) on X. Continuous homomorphisms are mappings f such that simultaneously $P^-(f)$ preserves the topology and f is compatible.

The above examples lead to the following definition. Let $F_1, \ldots, F_n$ be set functors (i.e. functors from the category of sets into itself), $\Delta_1, \ldots, \Delta_n$ types. We define a category $S((F_1, \Delta_1), \ldots, (F_n, \Delta_n))$ as follows: the objects are systems $(X, r_1, \ldots, \ldots, r_n)$, where r_i is a relational system of a type Δ_i on $F_i(X)$; a mapping $f : X \to Y$ is a morphism from $(X, r_1, \ldots, r_n)$ into $(Y, s_1, \ldots, s_n)$ if and only if $F_i(f)$ are r_is_i-compatible for covariant F_i's and s_ir_i-compatible for contravariant ones. (More exactly the morphisms are triples $((X, r_1, \ldots, r_n), f, (Y, s_1, \ldots, s_n))$; our simplified notation, however, will not lead to a confusion.)

Now, we see easily that the category of topological spaces is a full subcategory of $S(P^-, \{1\})$, the category of proximity spaces is a full subcategory of $S(P^+, \{2\})$ the category of uniform spaces is a full subcategory of $S(P^- \circ Q, \{1\})$, the category of merotopic spaces ([7]) is a full subcategory of $S(P^+ \circ P^+, \{1\})$, the category of bitopological spaces[1]) is a full subcategory of $S(P^- \circ P^+, \{1\})$, the category of topological groups is a full subcategory of $S((P^-, \{1\}), (I, \{3\}))$, where I is the identity functor, the category $\mathfrak{A}(\Delta)$ of all algebras of the type Δ and their homomorphisms is a full subcategory of $S(I, \bar{\Delta})$ (for $\bar{\Delta}$ see above), the category of all directed graphs and their graph-homomorphisms is exactly the category $S(I, \{2\})$ (it may be also described as $S(Q, \{1\})$. The category of topological spaces with open continuous mappings is a full subcategory of $S((P^-, \{1\}), (P^+, \{1\}))$ etc.

2. Representation

Let $\mathfrak{K}$, $\mathfrak{L}$ be categories. Full embedding of $\mathfrak{K}$ into $\mathfrak{L}$ is a one-to-one covariant functor Φ from $\mathfrak{K}$ into $\mathfrak{L}$, which maps $\mathfrak{K}$ onto a *full* subcategory of $\mathfrak{L}$. Hence, if a, b are objects in $\mathfrak{K}$, then the set of all morphisms from a into b is one-to-one mapped *onto* the set of all morphisms from $\Phi(a)$ into $\Phi(b)$. If a full embedding Φ from $\mathfrak{K}$ into $\mathfrak{L}$ exists, we write $\mathfrak{K} \rightrightarrows \mathfrak{L}$ and we say that $\mathfrak{K}$ is representable in $\mathfrak{L}$.

[1]) Bitopological spaces have been introduced by A. A. Ivanov.

We shall divide our discussion into two parts. First, we shall deal with the representation of categories in categories of algebras. The second part of this paragraph will be devoted to categories in which every boundable category is representable.

In [2] the following representation schema has been proved:

$$S(I, \Delta) \rightrightarrows S(I, \{2\}) \rightrightarrows \mathfrak{A}(\Delta'), \tag{1}$$

where $\Delta = \{\alpha_\beta \mid \beta < \gamma\}$ is arbitrary, $\Delta' = \{\alpha'_\beta \mid \beta < \gamma'\}$ is arbitrary such that $\Sigma\alpha'_\beta \geqq 2$.

As stated in the introduction, the question whether every concrete category is boundable remains open. It was shown in [4] that the answer may depend on set theory practicised (more exactly, it was shown there that in some rather odd set theories the answer is negative). However, we can prove that in the Gödel-Bernays set theory without measurable cardinals[2]) the categories of certain large family (see below) are boundable. Before stating the main theorem, we shall give some definitions.

Let us introduce the following notation. I is the identity functor. Let A be a set; Q_A is the functor associating with every X the set of all mappings X^A. If $f : X \to Y$ is a mapping, $Q_A(f) : Q_A(X) \to Q_A(Y)$ is defined by $Q_A(f)(\varphi) = f \circ \varphi$. Similarly, P_A^- associates with every set X the set A^X, with every $f : X \to Y$ the mapping $P_A^-(f) : : P_A^-(Y) \to P_A^-(X)$ defined by $P_A^-(f)(\varphi) = \varphi \circ f$.

K_A is defined as follows: $K_A(X) = X \times A$, $K_A(f) = f \times \mathrm{id}$ (i.e. $K_A(f)(x, a) = = (f(x), a)$). V_A is defined as follows: $V_A(X) = X \vee A$ $(= X \times \{0\} \cup A \times \{1\}$, i.e. a disjoint union of X and A), $V_A(f)(x, 0) = (f(x), 0)$ $V_A(f)(a, 1) = (a, 1)$. We recollect the definition of P^+ from § 1 and remark that Q is naturally equivalent with Q_2, P^- is naturally equivalent with P_2^-, I with Q_1 and K_1.

Now, we introduce some operations with set functors. The composition $F \circ G$ is obvious. If F, G are set functors of the same variance, $F \times G$ is defined by $(F \times G)(X) = F(X) \times G(X)$, $(F \times G)(f) = F(f) \times G(f)$. Similarly, the functor $F \vee G$ is defined. If F, G are functors of opposite variances, F^G is defined as follows: $F^G(X) = F(X)^{G(X)}$, $F^G(f)(\varphi) = F(f) \circ \varphi \circ G(f)$.

Metadefinition. *The constructive functors are defined recursively as follows*:

1. I, V_A, K_A, Q_A, P_A^-, P^+ *are constructive,*
2. *if* F, G *are constructive, so are* $F \circ G$, $F \times G$, $F \vee G$ *and* F^G *whenever the operations are defined,*
3. *if* F *is constructive and* G *naturally equivalent with* F, *then* G *is constructive.*

Now we can formulate the main theorem:

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[2]) This assumption may be easily weakened. But it is not known whether any assumption on measurable cardinals is needed at all.

Theorem. *Let $F_1, \ldots, F_n$ be constructive functors, $\Delta_1, \ldots, \Delta_n$ types. Then the category $S((F_1, \Delta_1), \ldots, (F_n, \Delta_n))$ is boundable. Consequently, every category which can be fully embedded into some $S((F_1, \Delta_1), \ldots, (F_n, \Delta_n))$ is boundable.*

Boundable categories are those, which can be represented in some category of algebras; on the other hand, we shall call binding such categories, in which every boundable category is representable. By (1), a category $\mathfrak{K}$ is binding if and only if the category $\mathfrak{A}(1, 1)$ is representable in $\mathfrak{K}$.

The categories $S(P^-, \{1\})$, $S(P^-, \{2\})$, $S(P^- \circ Q, \{1\})$, $S(P^+ \circ P^+, \{1\})$ etc., which appeared as examples in § 1 are binding. Of course, if we restrict ourselves to the objects satisfying axioms of topological spaces, uniform spaces, merotopic spaces etc., the binding property obviously vanishes. Let us dicsuss it on $S(P^-, \{1\})$. The full subcategory generated by those (X, t) with $(U_i \in t \Rightarrow \bigcup U_i \in t)$ is still binding. The full subcategory of topological spaces is not, due to "too many constants". We may ask whether it will be binding after some other choice of morphisms. It can be shown by a slight generalization of the method in [1] that the category of topological spaces with their local homeomorphisms is binding. A. B. Paalman-de Miranda proved, that this is not true for the category of Hausdorff spaces, however, the category of Hausdorff topological spaces, even of the compact ones, is binding, if we take as morphisms the quasi-local homeomorphisms ([8]). Many other choices of morphisms lead to binding categories.

3. Realization

The notion of concrete category is often used in the sense of category for which there exists a faithful functor into the category of all sets and all their mappings – the so called forgetful functor. In this paragraph, this notion will be used in a stronger sense; under a concrete category we shall mean a category, the objects of which are some sets (usually endowed by structures) and morphisms some mappings of these sets; thus, a concrete category is a category $\mathfrak{K}$ together with a fixed forgetful functor $\square$ on $\mathfrak{K}$. Let $(\mathfrak{K}, \square)$ and $(\mathfrak{K}', \square')$ be two concrete categories. Realization of $(\mathfrak{K}, \square)$ in $(\mathfrak{K}', \square')$ is a full embedding $\Phi : \mathfrak{K} \rightrightarrows \mathfrak{K}'$ such that $\square' \circ \Phi = \square$. We write then $\Phi : (\mathfrak{K}, \square) \rightrightarrows (\mathfrak{K}', \square')$. The existence of such a functor (realizability) is denoted by $(\mathfrak{K}, \square) \rightrightarrows (\mathfrak{K}', \square')$. If, for instance, the categories $(\mathfrak{K}, \square)$, $(\mathfrak{K}', \square')$ are categories of sets endowed by structures, the realizability means – roughly speaking – that structures of the first type may be canonically replaced by structures of the second one in such a way that the corresponding systems of morphisms are exactly the same systems of mappings between the underlying sets.

A simple example for illustration, showing a realization of partially ordered sets by means of topological spaces. Let $\leqq$ be a transitive relation on a set X. Define a topology $t(\leqq)$ on X as follows: $U \subset X$ is open if and only if $x \in U$, $y \leqq x$ imply $y \in U$. It is easy to see that, if $(X, \leqq)$, $(Y, \prec)$ are partially ordered sets, the continuous mappings from $(X, t(\leqq))$ into $(Y, t(\prec))$ are exactly the former isotone mappings.

The categories $S((F_1, \Delta_1), \ldots, (F_n, \Delta_n))$ and $\mathfrak{A}(\Delta)$ are always supposed to be endowed by the natural forgetful functor associating with an object $(X, r_1, \ldots, r_n)$ the set X etc. Therefore the symbols for forgetful functors will be omitted; we hope there is no danger of misunderstanding.

While from the point of view of representation all (with trivial exceptions) the categories $\mathfrak{A}(\Delta)$ were "almost the same", this is not the case with the realization. A necessary (and not sufficient) condition for $\mathfrak{A}(\Delta)$ being realizable in $\mathfrak{A}(\Delta')$ is, e.g., $\sup \Delta \leqq \sup \Delta'$. Hence, for instance, the category of grupoids is not realizable in any category of algebras with only unary operations, whatever the number of the unary operations may be.

A theorem on realization concerning "topology-like" categories has been proved by M. Katětov in [7]. Namely, it was shown that many categories discussed in topology are realizable in the category of merotopic spaces.

The following theorem holds for categories defined by functors:

Theorem. *Let $F_1, \ldots, F_n$ be constructive functors $\Delta_1, \ldots, \Delta_n$ types. Then there exist a natural number k and a set A such that*

$$S((F_1, \Delta_1), \ldots, (F_n, \Delta_n)) \rightrightarrows S((P^-)^k \circ V_A, \{1\}) .$$

We remark that there are intermediate questions between representation and realization which are of some interest. It may be shown e.g. that from some point of view the topological categories and the categories of relation differ less than the categories of algebras and categories of relational systems.

Proofs of the theorems had not been published yet. A special case of the theorem from § 2, covering e.g. the categories mentioned in § 1, has been proved in [3]. The present form of the theorem in § 2 follows from [3] and from the theorem in § 3 under the assumption that no measurable cardinal exists. The proofs of the other statements can be found in the papers refered to.

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