ON CONVERGENCE TOPOLOGIES

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A closure space (P, u) is a set P and a closure operator u satisfying the following three Kuratowski closure axioms: $u\emptyset = \emptyset$, $A \subset uA$ for each $A \subset P$ and $u(A \cup B) = uA \cup uB$ for each $A \subset P$ and $B \subset P$. The fourth axiom $(u(uA) = uA$ for each $A \subset P)$ need not be satisfied; therefore the closure topology need not be a topology.

A convergence $\mathfrak{L}$ on a set L is a set of pairs $(\{x_n\}, x)$ where $\{x_n\}$ is a sequence of points $x_n \in L$, $n \in N$, and $x \in L$, satisfying the Fréchet axioms

$(\mathscr{L}_0)$ If $(\{x_n\}, x) \in \mathfrak{L}$ and $(\{x_n\}, y) \in \mathfrak{L}$, then $x = y$.
$(\mathscr{L}_1)$ If $x_n = x$, $n \in N$, then $(\{x_n\}, x) \in \mathfrak{L}$.
$(\mathscr{L}_2)$ If $(\{x_n\}, x) \in \mathfrak{L}$ and $n_i < n_{i+1}$, $i \in N$, then $(\{x_{n_i}\}, x) \in \mathfrak{L}$.

The convergence $\mathfrak{L}$ is called a largest convergence and denoted by $\mathfrak{L}^*$ if it satisfies the Urysohn axiom

$(\mathscr{L}_3)$ If each subsequence $\{x_{n_i}\}$ of a sequence $\{x_n\}$ contains a subsequence $\{x_{n_{i_j}}\}$ such that $(\{x_{n_{i_j}}\}, x) \in \mathfrak{L}$, then $(\{x_n\}, x) \in \mathfrak{L}$.

Let $\mathfrak{L}$ be a convergence on a set L. Define mapping λ on the family of all subsets of L into itself as follows: If $A \subset L$ and $x \in L$, then $x \in \lambda A$ if there is a sequence $\{x_n\}$ such that $(\{x_n\}, x) \in \mathfrak{L}$ and $\bigcup_{n=1}^{\infty} x_n \subset A$. The mapping λ is a closure topology for L and it is called a convergence topology. The closure space (L, λ) is called a convergence space and denoted by $(L, \mathfrak{L}, \lambda)$ ([2]).

The family of all continuous functions on a convergence space $(L, \mathfrak{L}, \lambda)$ to the closed interval $\langle 0,1 \rangle$ will be denoted by $\mathfrak{F}(L)$.

J. NovÁk defined [1] the notion of sequential regularity as follows:

A convergence space $(L, \mathfrak{L}, \lambda)$ is sequentially regular if for each point $x \in L$ and for each sequence $\{x_n\}$ such that $(\{x_{n_i}\}, x) \notin \mathfrak{L}$ for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$ there is a function $f \in \mathfrak{F}(L)$ such that the sequence $\{f(x_n)\}$ does not converge to $f(x)$.

We shall say that a convergence space $(L, \mathfrak{L}, \lambda)$ has the property (P) if for each $x \in L$, $y \in L$, $x \neq y$, there is $f \in \mathfrak{F}(L)$ such that $f(x) \neq f(y)$.

Let $(L, \mathfrak{L}, \lambda)$ be a convergence space. Let $\hat{\mathfrak{L}}$ be the set of all pairs $(\{x_n\}, x)$ such that the sequence $\{f(x_n)\}$ converges to $f(x)$ for each $f \in \mathfrak{F}(L)$. The set $\hat{\mathfrak{L}}$ is a convergence if and only if the convergence space $(L, \mathfrak{L}, \lambda)$ has the property (P).

Definition 1. Let $(L, \mathfrak{L}, \lambda)$ be a convergence space with the property (P). The convergence topology induced by the convergence $\hat{\mathfrak{L}}$ will be called a *sequentially regular modification of* λ and denoted $\hat{\lambda}$.

The following statements clarify the relation between λ and $\hat{\lambda}$:

The convergence space $(L, \hat{\mathfrak{L}}, \hat{\lambda})$ is sequentially regular. The sequentially regular modification $\hat{\lambda}$ of λ is the weakest[1]) of all sequentially regular convergence topologies stronger than λ. Consequently $\lambda = \hat{\lambda}$ if and only if the space $(L, \mathfrak{L}, \lambda)$ is sequentially regular.

Definition 2. Let $(L, \mathfrak{L}, \lambda)$ be a convergence space with the property (P). Then there exists the weakest of all completely regular separated topologies stronger than λ. It will be called a completely regular modification of λ and denoted $\tilde{\lambda}$.

Definition 3. Let (P, u) be a separated closure space. Define a convergence $\mathfrak{P} : (\{x_n\}, x) \in \mathfrak{P}$ if for each neighbourhood $U(x)$ of x we have $x_n \in U(x)$ for nearly all $n \in N$. The convergence space $(P, \mathfrak{P}, \pi)$ will be said to be associated with (P, u).

The following statements are true: $\lambda < \hat{\lambda} < \tilde{\lambda}$ and $\tilde{\hat{\lambda}} = \tilde{\lambda}$. The convergence space $(L, \hat{\mathfrak{L}}, \hat{\lambda})$ is associated with the space $(L, \tilde{\lambda})$. If the convergence space $(L, \mathfrak{L}, \lambda)$ is sequentially regular, then $\lambda = \hat{\lambda} < \lambda^{\omega_1} < \tilde{\lambda}$ where λ^{ω_1} denotes the topological modification of λ [2]).

Now we can characterize the class of sequentially regular spaces.

Theorem 1. *The class of sequentially regular spaces whose convergences are largest coincides with the class of convergence spaces associated with completely regular separated spaces.*

Consider the class **P** of completely regular separated spaces whose topology is a completely regular modification of some convergence topology. The class **P** is no hereditary and productive. If (P, u) is a completely regular separated space and $(P, \mathfrak{P}, \pi)$ is the convergence space associated with (P, u) then (P, u) belongs to **P** iff $\tilde{\pi} = u$.

Theorem 2. *A completely regular separated space (P, u) belongs to* **P** *if and only if the following condition is satisfied: a function f on P to $\langle 0,1 \rangle$ is continuous on (P, u) iff $\lim f(x_n) = f(x)$ whenever for each neighbourhood $U(x)$ of x we have $x_n \in U(x)$ for nearly all $n \in N$.*

The notion of sequential evelope was defined ([1]) as follows:

A sequentially regular convergence space $(S, \mathfrak{S}, \sigma)$ is a sequential envelope $\sigma(L)$ of a convergence space $(L, \mathfrak{L}, \lambda)$ if the following conditions are satisfied:

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[1]) If μ and ν are closure topologies for L, then μ is weaker than ν if $\mu A \subset \nu A$ for each $A \subset L$.

[2]) A topological modification λ^{ω_1} of a convergence topology λ is the weakest of all topologies stronger than λ.

(σ_0) $(L, \mathfrak{L}, \lambda)$ is a subspace of $(S, \mathfrak{S}, \sigma)$.

(σ_1) $S = \sigma^{\omega_1} L$.

(σ_2) Each function $f \in \mathfrak{F}(L)$ has an extension $\tilde{f} \in \mathfrak{F}(S)$.

(σ_3) There is no sequentially regular space $(S', \mathfrak{S}', \sigma')$ containing $(S, \mathfrak{S}, \sigma)$ as a proper subspace and satisfying (σ_1) and (σ_2) with respect to L.

It has been pointed out ([1]) that the definition of a sequential envelope is similar to that of the Stone-Čech compactification. The following theorem states their relation in an explicit form.

Theorem 3. *Let $(L, \mathfrak{L}, \lambda)$ be a sequentially regular space, let $\tilde{\lambda}$ be the sequentially regular modification of λ and let (P, u) be the Stone-Čech compactification of $(L, \tilde{\lambda})$. Let $(P, \mathfrak{P}, \pi)$ be the convergence space associated with (P, u). Then $\sigma(L) = \pi^{\omega_1} L$.*

Theorem 4. *Let $(L, \mathfrak{L}, \lambda)$ be a sequentially regular space. If $(L, \mathfrak{L}, \lambda)$ is countably compact or if $(L, \tilde{\lambda})$ is a normal space, then $\sigma(L) = L$.*

However, neither of these two conditions is also necessary for the statement $\sigma(L) = L$. On the other hand there exist convergence rings of sets $\boldsymbol{M}$ such that $\sigma(\boldsymbol{M}) \neq \boldsymbol{M}$.

Remark. The paper with proofs will appear in the Czechoslovak Mathematical Journal in 1967.

References

[1] *J. Novák:* On the sequential envelope. General Topology, Proc. of the Symp. Prague 1961, Praha 1962, 292—294.

[2] *J. Novák:* On convergence spaces and their sequential envelopes. Czech. Math. J. *15* (90) (1965), 74—100.