ON URYSOHN'S LEMMA

C. H. DOWKER and DONA PAPERT

London

In this paper we show how a well known non-tautological theorem of point-set topology can be proved in frame theory, that is in topology without points. The results are not new but were proved in the unpublished Cambridge dissertation: Dona Papert, Lattices of functions, measures and point sets, 1958.

A partially ordered set is a set L with a relation $\leqq$, such that

1) if $a \leqq b$ and $b \leqq c$ then $a \leqq c$, and
2) if $a \leqq b$ and $b \leqq a$ then $a = b$,

A complete lattice is a partially ordered set such that

3) every subset A of L has a least upper bound.

The least upper bound is unique and is usually called the join of A and written $\bigvee A$ or, in terms of elements, $\bigvee a_\alpha$ or $a_1 \vee a_2$. Let $1 = \bigvee L$; then 1 is the greatest element of L. Let $0 = \bigvee \emptyset$, where $\emptyset$ is the empty set; then 0 is the least element of L. The operation $\vee$ is associative and commutative, for the join depends on the set A, not on the arrangement of its elements.

If B is the set of lower bounds of A, each $a \in A$ is an upper bound of B and hence $\bigvee B \leqq a$. Thus $\bigvee B$ is a lower bound of A. This greatest lower bound of A is called the meet of A and written $\bigwedge A$, $\bigwedge a_\alpha$ or $a_1 \wedge a_2$. Clearly $\bigwedge L = 0$ and $\bigwedge \emptyset = 1$.

The topology T of a space X, that is the set of all open sets of X, is a complete lattice with the relation $\subseteq$. For any family $\{G_\alpha\}$ of open sets, the join $\bigvee G_\alpha$ is the union $\bigcup G_\alpha$ and the meet $\bigwedge G_\alpha$ is the interior of the intersection $\bigcap G_\alpha$, thus $G_1 \wedge G_2 = G_1 \cap G_2$. The elements 0 and 1 of T are $\emptyset$ and X.

A *frame* is a complete lattice satisfying the distributive law

4) $a \wedge \bigvee b_\alpha = \bigvee a \wedge b_\alpha$.

In particular $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. Also we have $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, for $(a \vee b) \wedge (a \vee c) = ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c) = a \vee (a \wedge c) \vee (b \wedge c) = a \vee (b \wedge c)$. From 4) by commutativity we have $(\bigvee a_\alpha) \wedge b = \bigvee (a_\alpha \wedge b)$. Applying 4) again gives $\bigvee a_\alpha \wedge \bigvee b_\beta = \bigvee (a_\alpha \wedge \bigvee b_\beta) = \bigvee_\alpha \bigvee_\beta a_\alpha \wedge b_\beta$, and, by induction, $\bigvee a_\alpha \wedge \bigvee b_\beta \wedge \ldots \wedge \bigvee c_\gamma = \bigvee_\alpha \bigvee_\beta \cdots \bigvee_\gamma a_\alpha \wedge b_\beta \wedge \ldots \wedge c_\gamma$. The topology T of a space X is clearly a frame.

If L and M are frames, a function $\varphi : L \to M$ is called a *frame map*, or simply a map, if $\varphi \bigvee a_\alpha = \bigvee \varphi a_\alpha$ for each family $\{a_\alpha\}$ and $\varphi \bigwedge a_i = \bigwedge \varphi a_i$ for each finite family $\{a_i\}$. In particular, when the families are empty, we have $\varphi 0_L = 0_M$ and $\varphi 1_L = 1_M$.

Let X_1, X_2 be spaces with topologies T_1, T_2, and let $f : X_1 \to X_2$ be a continuous function. For each $G \in T_2$, $f^{-1}G \in T_1$. Also $f^{-1} \bigvee G_\alpha = f^{-1} \bigcup G_\alpha = \bigcup f^{-1} G_\alpha = \bigvee f^{-1} G_\alpha$, and, for finite families $\{G_i\}$, $f^{-1} \bigwedge G_i = f^{-1} \bigcap G_i = \bigcap f^{-1} G_i = \bigwedge f^{-1} G_i$. Thus $f^{-1} : T_2 \to T_1$ is a frame map. We shall now show that all frame maps of topologies of Hausdorff spaces are obtained thus from continuous functions.

Theorem 1. *If X_1, X_2 are spaces with topologies T_1, T_2, if X_2 is a Hausdorff space and if $\varphi : T_2 \to T_1$ is a frame map, there exists a unique continuous function $f : X_1 \to X_2$ such that $f^{-1} = \varphi$.*

Proof. For any point $x \in X_1$, let G be the union of all open sets G_α of X_2 for which $x \notin \varphi G_\alpha$. Then $\varphi G = \varphi \bigcup G_\alpha = \bigcup \varphi G_\alpha$, so $x \notin \varphi G$. Thus G is the greatest open set of X_2 for which $x \notin \varphi G$.

Since $\varphi 1 = 1$, that is $\varphi X_2 = X_1$, and since $x \in X_1$, hence $G \neq X_2$. Let $y \in X_2 \setminus G$. If z is any other point of the Hausdorff space X_2, there are disjoint open sets U, V with $y \in U$, $z \in V$. Then $\varphi U \cap \varphi V = \varphi(U \cap V) = \varphi \emptyset = \emptyset$. Then $x \in \varphi U$, $x \notin \varphi V$, so $V \subseteq G$ and $z \in G$. Thus there is only one point $y \in X_2 \setminus G$.

For each $x \in X_1$, let $f(x)$ be the point of X_2 not in $\max\{G : x \notin \varphi G\}$. Then for H open in X_2, $f(x) \in H$ if and only if $x \in \varphi H$; that is $f^{-1}H = \varphi H$. Thus $f^{-1}H$ is open, so f is continuous. And we have $f^{-1} = \varphi$.

If $g : X_1 \to X_2$ is another continuous function, choose $x \in X_1$ for which $g(x) \neq f(x)$. Let $H = X_2 \setminus (g(x))$. Then $x \in f^{-1}H = \varphi H$ but $x \notin g^{-1}H$. Thus $g^{-1} \neq \varphi$. This completes the proof.

A *base* B of a frame L is a subset of L such that every element of L is a join of elements of B.

Theorem 2. *Let L and M be frames, let B be a base of L and let $\varphi : B \to M$ be a function such that if $\{b_i\}$ is finite and $\bigwedge b_i \leqq \bigvee c_\alpha$ then $\bigwedge \varphi b_i \leqq \bigvee \varphi c_\alpha$. Then φ extends to a frame map $\mu : L \to M$.*

(When the family $\{b_i\}$ is empty, the hypothesis states that if $1 = \bigvee c_\alpha$ then $1 = \bigvee \varphi c_\alpha$. In particular $\bigvee_{c \in B} \varphi c = 1$.)

Proof. For $h \in L$ we define $\mu h = \bigvee_{b \in B, b \leqq h} \varphi b$. If $b \leqq c$ in B then $\varphi b \leqq \varphi c$. Thus for $c \in B$ we have $\mu c = \bigvee_{b \leqq c} \varphi b = \varphi c$. Thus μ is an extension of φ.

If $h \leqq k$ then $\mu h = \bigvee_{b \leqq h} \varphi b \leqq \bigvee_{b \leqq k} \varphi b$; hence $\mu h \leqq \mu k$.

For a finite non-empty family $\{h_i\}$, $i = 1, \ldots, n$, we have

$$\bigwedge \mu h_i = \bigvee_{a \leqq h_1} \varphi a \wedge \bigvee_{b \leqq h_2} \varphi b \wedge \ldots \wedge \bigvee_{c \leqq h_n} \varphi c = \bigvee_{a \leqq h_1} \ldots \bigvee_{c \leqq h_n} \varphi a \wedge \ldots \wedge \varphi c .$$

Since $a \wedge \ldots \wedge c \leqq \bigwedge h_i = \bigvee_{b \leqq \bigwedge h_i} b$, hence by hypothesis

$$\varphi a \wedge \ldots \wedge \varphi c \leqq \bigvee_{b \leqq \bigwedge h_i} \varphi b = \mu \bigwedge h_i .$$

Thus $\bigwedge \mu h_i \leqq \mu \bigwedge h_i$. But since $\bigwedge h_i \leqq h_i$, $\mu \bigwedge h_i \leqq \mu h_i$, and hence $\mu \bigwedge h_i \leqq \bigwedge \mu h_i$. Therefore $\mu \bigwedge h_i = \bigwedge \mu h_i$.

In case $\{h_i\}$ is empty this is still true, namely $\mu 1 = 1$, for $\mu 1 = \bigvee_{b < 1} \varphi b = 1$.

For any family $\{h_\alpha\}$ we have $\mu \bigvee h_\alpha = \bigvee_{b \leqq \bigvee h_\alpha} \varphi b$. When $b \leqq \bigvee h_\alpha = \bigvee_\alpha \bigvee_{c \in B, c \leqq h_\alpha} c$, then $\varphi b \leqq \bigvee_\alpha \bigvee_{c \leqq h_\alpha} \varphi c = \bigvee_\alpha \mu h_\alpha$. Hence $\mu \bigvee h_\alpha \leqq \bigvee \mu h_\alpha$. But since $\bigvee h_\alpha \geqq h_\alpha$, $\mu \bigvee h_\alpha \geqq \mu h_\alpha$ for each α and hence $\mu \bigvee h_\alpha \geqq \bigvee \mu h_\alpha$. Thus in each case $\mu \bigvee h_\alpha = \bigvee \mu h_\alpha$.

Thus μ is a frame map, as was to be shown.

A frame L is called *normal* if, whenever $u \vee v = 1$, there exist g, h such that

$$g \vee v = 1 , \quad u \vee h = 1 , \quad g \wedge h = 0 .$$

Clearly the topology of a space X is normal if and only if X is a normal space.

Theorem 3. *If L is a normal frame and $u \vee v = 1$ in L there exists a frame map $\mu : T_R \to L$, where T_R is the topology of the real line R, such that $\mu(R \setminus (0)) \leqq u$, $\mu(R \setminus (1)) \leqq v$.*

Proof. Let Q be the set of rational numbers. We shall construct $g_p, h_p \in L$ for $p \in Q$ so that $g_p \wedge h_p = 0$ and, if $p < q$, $g_p \vee h_q = 1$. When they are thus defined for p and q with $p < q$ we have $h_p = h_p \wedge 1 = h_p \wedge (g_p \vee h_q) = h_p \wedge h_q$, so $h_p \leqq h_q$, and also $g_q = g_q \wedge 1 = g_q \wedge (g_p \vee h_q) = g_q \wedge g_p$ so $g_p \geqq g_q$.

The rationals between 0 and 1 are countable; call them $r_1, r_2, \ldots$ Let Q_n consist of all the rationals $\leqq 0$ or $\geqq 1$ and $r_1, r_2, \ldots, r_n$. For $p \in Q_0$ we define g_p, h_p as follows: $g_p = 1$, $h_p = 0$ for $p < 0$; $g_0 = u$, $h_0 = 0$; $g_1 = 0$, $h_1 = v$, $g_p = 0$, $h_p = 1$ for $p > 1$.

Suppose g_p, h_p have been defined for $p \in Q_n$. We now define g_r, h_r for $r = r_{n+1}$. Take the greatest $p \in Q_n$ with $p < r$ and the least $q \in Q_n$ with $q > r$. Then $p < q$ and $g_p \vee h_q = 1$. By normality there exist g_r, h_r for which $g_r \vee h_q = 1$, $g_p \vee h_r = 1$, $g_r \wedge h_r = 0$. If $s \in Q_{n+1}$ and $s < r$ then $s \leqq p$, $g_s \geqq g_p$ and $g_s \vee h_r = 1$. If $s > r$ then $s \geqq q$, $h_s \geqq h_q$ and $g_r \vee h_s = 1$. Thus g_s, h_s with the required properties are defined for all $s \in Q_{n+1}$. Hence by induction they can be defined for all $s \in Q$.

Take the base $B \subset T_R$ consisting of all open intervals (x, y) with $x < y$. The function $\varphi : B \to L$ is defined by

$$\varphi(x, y) = \bigvee_{x < p < q < y} g_p \wedge h_q = \bigvee_{x < p} g_p \wedge \bigvee_{q < y} h_q .$$

Let (x_i, y_i), $i = 1, \ldots, n$ be a non-empty finite family of intervals, and let (x_α, y_α) be a family of intervals such that $\bigcap(x_i, y_i) \subseteq \bigcup(x_\alpha, y_\alpha)$. Then

$$\begin{aligned}\bigwedge\varphi(x_i, y_i) &= \Big(\bigvee_{x_1<p_1<q_1<y_1} g_{p_1} \wedge h_{q_1}\Big) \wedge \ldots \wedge \Big(\bigvee_{x_n<p_n<q_n<y_n} g_{p_n} \wedge h_{q_n}\Big)\\ &= \bigvee \ldots \bigvee g_{p_1} \wedge h_{q_1} \wedge \ldots \wedge g_{p_n} \wedge h_{q_n}\\ &= \bigvee \ldots \bigvee g_{\max p} \wedge h_{\min q}\\ &= \bigvee_{\max x_i<p<q<\min y_i} g_p \wedge h_q\\ &= \varphi \bigcap(x_i, y_i)\,.\end{aligned}$$

For any rational numbers p, q such that $\max x_i < p < q < \min y_i$, the compact interval $[p, q]$ is contained in $\bigcup_\alpha(x_\alpha, y_\alpha) = \bigcup_\alpha \bigcup_{x_\alpha<r<s<y_\alpha} (r, s)$ for r, s rational. Hence $[p, q]$ is contained in some finite number of these intervals (r, s), so the open interval (p, q) is a finite union $\bigcup_j(r_j, s_j)$ of such intervals. We may assume that no (r_j, s_j) can be omitted from the union and that (r_j, s_j) overlaps (r_{j+1}, s_{j+1}).

If $r < t < s < u$ we have $(g_r \wedge h_s) \vee (g_t \wedge h_u) = (g_r \vee g_t) \wedge (g_r \vee h_u) \wedge \wedge (h_s \vee g_t) \wedge (h_s \vee h_u) = g_r \wedge h_u$. Hence $g_p \wedge h_q = \bigvee g_{r_j} \wedge h_{s_j} \leqq \bigvee\varphi(x_\alpha, y_\alpha)$. Hence $\bigwedge\varphi(x_i, y_i) \leqq \bigvee\varphi(x_\alpha, y_\alpha)$.

If $\bigcup(x_\alpha, y_\alpha) = R$ then $(-2, 3) \subseteq \bigcup(x_\alpha, y_\alpha)$ and hence $1 = g_{-1} \wedge h_2 \leqq \leqq \varphi(-2, 3) \leqq \bigvee\varphi(x_\alpha, y_\alpha)$. Thus $\bigwedge\varphi(x_i, y_i) \leqq \bigvee\varphi(x_\alpha, y_\alpha)$ even when the family (x_i, y_i) is empty. Therefore φ extends to a frame map $\mu : T_R \to L$.

If $x < y < 0$ then $\varphi(x, y) = 0$. If $0 < x < y$ then for $x < p < q < y$ we have $g_p \wedge h_q \leqq g_0 = u$, and hence $\varphi(x, y) \leqq u$. Hence $\mu(R \setminus (0)) = \bigvee_{0\notin(x,y)} \varphi(x, y) \leqq u$.

If $x < y < 1$ then for $x < p < q < y$ we have $g_p \wedge h_q \leqq h_1 = v$ and hence $\varphi(x, y) \leqq v$. If $1 < x < y$ then $\varphi(x, y) = 0$. Hence $\mu(R \setminus (1)) = \bigvee_{1\notin(x,y)} \varphi(x, y) \leqq v$. This completes the proof.

Theorem 4 (Urysohn). *If E, F are disjoint closed sets of a normal space X there is a continuous real function $f : X \to R$ such that $f(x) = 0$ when $x \in E$ and $f(x) = 1$ when $x \in F$.*

Proof. Let $U = X \setminus E$, $V = X \setminus F$; then $U \cup V = X$. By Theorem 3 there is a map $\mu : T_R \to T_X$, where T_X is the topology of X, such that $\mu(R \setminus (0)) \subseteq U$, $\mu(R \setminus (1)) \subseteq V$. By Theorem 1, since R is a Hausdorff space, there is a continuous function $f : X \to R$ such that $f^{-1} = \mu$. Since $f^{-1}(R \setminus (0)) \subseteq U, f(E) \subseteq (0)$. And since $f^{-1}(R \setminus (1)) \subseteq V$, $f(F) \subseteq (1)$. This completes the proof.

Reference

[1] *C. H. Dowker and Dona Papert:* Quotient frames and subspaces. Proc. London Math. Soc. *16* (1966), 275—296.