BOREL SUBSETS OF METRIC SEPARABLE SPACES

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In this note, we shall study a connection between the following two sentences:

(L) $2^{\aleph_0} = 2^{\aleph_1}$ (Luzin hypothesis),

(B) In every separable non-denumerable metric space, there is a subset which is not a Borel set.

It is well known that the negation of (L) implies (B) (see, e.g., [4], p. 253). In the following we shall prove the consistency of (L) and (B) with the axioms of set theory. That gives partial solution of a problem posed by prof. Kuratowski ([4], p. 254).

The terminology and notations used are those of [2] and [3]. We remind the reader of some notions and facts. A class M is called perfect iff

(i) M is closed under fundamental operations $\mathfrak{F}_1, \ldots, \mathfrak{F}_8$, i.e.,

$$(\forall x)(\forall y)(x, y \in M \rightarrow \mathfrak{F}_i(x, y) \in M), \quad i = 1, 2, \ldots, 8,$$

(ii) M is almost universal, i.e.,

$$(\forall z)(z \subseteq M \rightarrow (\exists x)\ (x \in M \,\&\, z \subseteq x)),$$

(iii) M is complete, i.e.,

$$(\forall x)(x \in M \rightarrow x \subseteq M).$$

Every perfect class determines a model of the theory $\sum$ (axioms A–D, see [3], p. 335).

From the topology of metric spaces, it is well known that (B) is equivalent to the following sentence:

(C) Every subset x of the Hilbert cube J^{ω_0} of power $\aleph_1$ (J is the open unit interval) contains a subset $y \subseteq x$ which is not a Borel set in x.

We may suppose that $J \subseteq \mathfrak{P}(\omega_0)$ (i.e., every real number x, $0 < x < 1$, is a subset of ω_0). Let $G_0(X)$ denote the open basis of a separable metric space X. We define

$$y \in G_\alpha(X) \equiv (\exists f)(\mathfrak{D}(f) \subseteq \omega_0 \,\&\, \mathfrak{W}(f) \subseteq \bigcup_{\xi \in \alpha} G_\xi(X) \,\&\, y = \bigcup \mathfrak{W}(f)) \quad \text{for} \quad \alpha \text{ even},$$

$$y \in G_\alpha(X) \equiv (\exists f)(\mathfrak{D}(f) \subseteq \omega_0 \,\&\, \mathfrak{W}(f) \subseteq \bigcup_{\xi \in \alpha} G_\xi(X) \,\&\, y = \bigcap \mathfrak{W}(f)) \quad \text{for} \quad \alpha \text{ odd}.$$

The set of Borel subsets of X is

$$\mathfrak{B}(X) = \bigcup_{\xi \in \omega_1} G_\xi(X)$$

(see [1], [4]).

The absoluteness of a notion is defined in [2] and [3].

Lemma. *Let M be a perfect class. If $\mathfrak{P}(\omega_0)$ and ω_1 are absolute relative to M, then $\mathfrak{B}(J^{\omega_0})$ is absolute relative to M (thus, $\mathfrak{B}(J^{\omega_0}) \subseteq M$).*

Proof. Let $\mathfrak{P}(\omega_0)$, ω_1 be absolute relative to M. It is easy to see that J, $(\omega_0^{\omega_0})^{\omega_0}$ are absolute too. We define $G_0(J)$ as the set of all open intervals (a, b), where $0 \leqq a \leqq b \leqq 1$, a, b are rational numbers. $G_0(J)$ is absolute relative to M. Now, we can define

$$x \in G_0(J^{\omega_0}) \equiv (\exists f) (\mathfrak{D}(f) = \omega_0 \,\&\, \mathfrak{W}(f) \subseteq G_0(J) \,\&\, \overline{\overline{\{n; f(n) \neq J\}}} < \aleph_0 \,\&$$
$$\& \, (\forall y) (y \in J^{\omega_0} \rightarrow .\; y \in x \equiv (\forall n) (y(n) \in f(n)))) .$$

Evidently, $G_0(J^{\omega_0})$ is absolute relative to M. We shall proceed by induction. Let $G_\xi(J^{\omega_0})$ be absolute for $\xi \in \alpha$. Using the absoluteness of a sum, we have $(\bigcup_{\xi \in \alpha} G_\xi(J^{\omega_0}))_M = \bigcup_{\xi \in \alpha} G_\xi(J^{\omega_0})$. Moreover, we have $(G_\alpha(J^{\omega_0}))_M \subseteq G_\alpha(J^{\omega_0})$. Let $x \in G_\alpha(J^{\omega_0})$. If α is even, there is $f \in (\bigcup_{\xi \in \alpha} G_\xi(J^{\omega_0}))^{\omega_0}$ such that $x = \bigcup \mathfrak{W}(f)$. In the model defined by M, there is an one-to-one mapping g of the set $\bigcup_{\xi \in \alpha} G_\xi(J^{\omega_0})$ onto $\omega_0^{\omega_0}$. Let $h = g \circ f$. Since $h \in (\omega_0^{\omega_0})^{\omega_0}$, then $h \in M$ and $f \in M$ $(f = g^{-1} \circ h)$. Therefore, $\bigcup \mathfrak{W}(f) = x \in M$, i.e., $G_\alpha(J^{\omega_0}) = (G_\alpha(J^{\omega_0}))_M$. The argument is similar for α odd. Using the absoluteness of ω_1 we have

$$(\mathfrak{B}(J^{\omega_0}))_M = \mathfrak{B}(J^{\omega_0}) .$$

Let Λ denote a particular ordinal number greater than zero (see [3], p. 321). From [6] (for Zermelo-Fraenkel set theory from [5]) the consistency of the following assumptions follows:

(1) $2^{\aleph_0} = \aleph_{\Lambda \dot{+} 1}$, $2^{\aleph_1} = \aleph_{\Lambda \dot{+} 2}$,

(2) cardinal numbers are absolute.

In the following, we shall work in the theory $\sum^*$ with axioms (1) and (2).

Let k, f, g denote functions with properties:

$$\mathfrak{k}(k, 0, \omega_{\Lambda \dot{+} 1}) , \quad k \in L \quad (\text{see [3], } p. \text{ 352}) ,$$

$$\mathrm{Un}_2(f) \,\&\, \mathfrak{D}(f) = \omega_{\Lambda \dot{+} 1} \,\&\, \mathfrak{W}(f) = \mathfrak{P}(\omega_0) ,$$

$$\mathrm{Un}_2(g) \,\&\, \mathfrak{D}(g) = \omega_{\Lambda \dot{+} 2} \,\&\, \mathfrak{W}(g) = \mathfrak{P}(\omega_1) .$$

The existence of k, f, g follows from (1) and (2). Now, we define

$$\begin{aligned} h_0(\eta) &= f(\eta - 1) && \text{for} \quad \eta \in K_{\mathrm{I}} \cap \omega_{\mathbf{A}+1}\,, \\ h_0(\eta) &= 0 && \text{for} \quad \eta \in K_{\mathrm{II}} \cap \omega_{\mathbf{A}+1}\,, \\ h_\xi(\eta) &= f(\eta - 1) && \text{for} \quad \eta \in K_{\mathrm{I}} \cap \omega_{\mathbf{A}+1}\,, \\ h_\xi(\eta) &= h_{\eta+1}(\eta) && \text{for} \quad \eta \in K_{\mathrm{II}} \cap \omega_{\mathbf{A}+1}\,, \quad \eta + 1 < \xi\,, \\ h_\xi(\eta) &= 0 && \text{for} \quad \eta \in K_{\mathrm{II}} \cap \omega_{\mathbf{A}+1} - \xi\,, \\ h_\xi(\eta) &= a && \text{for} \quad \eta + 1 = \xi\,, \quad \eta \in K_{\mathrm{II}} \cap \omega_{\mathbf{A}+1}\,, \end{aligned}$$

where $a = g(\lambda)$ and λ is the least ordinal for which

$$g(\lambda) \notin \mathfrak{G}(h_{\xi-1}, k, \omega_{\mathbf{A}+1})''\, \omega_{\mathbf{A}+2}\,.$$

Let $h(\xi) = h_{\xi+1}(\xi)$.
The definition of the perfect class $\mathfrak{M}(h, k, \omega_{\mathbf{A}+1})$ is given in [3].

Theorem. *In the model defined by the perfect class $\mathfrak{M}(h, k, \omega_{\mathbf{A}+1})$ the following assertions hold*:

(i) $2^{\aleph_0} = 2^{\aleph_1} = \aleph_{\mathbf{A}+1}$,
(ii) *cardinal numbers are those of the whole theory*,
(iii) $(\forall x)\,(x \subseteq J^{\omega_0} \,\&\, \bar{\bar{x}} = \aleph_1 . \rightarrow \mathfrak{B}(x) \neq \mathfrak{P}(x))\,.$

Proof. (i) and (ii) follow from definitions and [3] immediately. We shall prove (iii). Let $x \subseteq J^{\omega_0}$, i.e., $x \subseteq \mathfrak{P}(\omega_0)^{\omega_0}$, $\bar{\bar{x}} = \aleph_1$. The definition of the function h_ξ implies the existence of an ordinal $\xi_0 \in \omega_{\mathbf{A}+1}$ for which

$$x \subseteq \mathfrak{G}(h, k, \omega_{\mathbf{A}+1})''\, \xi_0\,.$$

By 4. 10. 3 from [3], there is an $\xi_1 \in \omega_{\mathbf{A}+1}$ such that

$$x \in \mathfrak{M}(h, k, \xi_1) \subseteq \mathfrak{M}(h_{\xi_1}, k, \omega_{\mathbf{A}+1})\,.$$

There is a one-to-one mapping $g \in \mathfrak{M}(h_{\xi_1}, k, \omega_{\mathbf{A}+1})$ of x onto ω_1 (since cardinals are absolute). Let $a \notin \mathfrak{M}(h_{\xi_1}, k, \omega_{\mathbf{A}+1})$, $a \subseteq \omega_1$ (it suffices to define $a = h(\eta)$, where η is the first limit number greater than ξ_1).

Let us suppose that $g^{-1}(a)$ is a Borel subset of x, i.e., $g^{-1}(a) = x \cap y$, $y \in \mathfrak{B}(J^{\omega_0})$. Using lemma and the definition of h, we have $y \in \mathfrak{M}(h_{\xi_1}, k, \omega_{A+1})$, thus $a \in \mathfrak{M}(h_{\xi_1}, k, \omega_{\mathbf{A}+1})$ – a contradiction. Hence, $g^{-1}(a)$ is not a Borel subset of x and our proof is complete.

Using well-known facts, we obtain

Metatheorem. *Let φ be an elementary formula of the theory $\sum_0$, for which* $\vdash_{\Sigma_0} (\forall x)\,(\varphi(x) \rightarrow x \in On \,\&\, x \neq 0) \,\&\, (\exists!\, x)\, \varphi(x)$ ($\sum_0$ *is the theory with axioms* $A - C$).

If $\sum_0$ is consistent, then the theory $\sum^*$ with axioms

(i) $(\forall x)(\varphi(x) \rightarrow 2^{\aleph_0} = 2^{\aleph_1} = \aleph_{x+1})$,

(ii) In every non-denumerable metric separable space, there is a subset which is not Borel,

is consistent.

References

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