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THE SIMPLE DIMENSION OF A TOPOLOGICAL SPACE

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1. Introduction. Given a set S and any topology $\mathscr{T}$ for S, such that the space $C(S, \mathscr{T})$ of all real-valued continuous functions on $(S, \mathscr{T})$ separates the points of S, there is a Tikhonov (completely regular and T_1-) topology $\mathscr{S}$ for S, with $\mathscr{T} \supset \mathscr{S}$, such that $C(S, \mathscr{T}) = C(S, \mathscr{S})$. For any set $D(S)$ of functions which separates the points of S, there is a unique smallest Tikhonov topology $\mathscr{S}$ for S such that $D(S) \subset C(S, \mathscr{S})$. (See Chapter 3 of [4], and [12].)

The evaluation map which is defined by $f(s) = \{x(s)\}$, over $x \in C(S, \mathscr{S})$, is a homeomorphism of $(S, \mathscr{S})$ into a Cartesian product of real lines with $C(S, \mathscr{S})$ as index. In case $\mathscr{S}$ is a proper subtopology of $\mathscr{T}$, the same evaluation map is a continuous, but non-homeomorphic, injection of $(S, \mathscr{T})$ into the Cartesian product.

The *linear embedding characteristic* of a Tikhonov space S is the cardinal of the smallest index J, such that the corresponding evaluation map into a product of lines, with J as index, is injective. The *simple dimension at* $s \in S$ similarly is the cardinal of the smallest index such that there is an open neighborhood O_s of s such that the evaluation map is an injection of O_s. The *simple dimension* of S is the supremum of the simple dimensions over $s \in S$.

In this work we propose to investigate relationships between the linear embedding characteristic, simple dimension, and the covering and inductive topological dimensions. It is shown in [7] that for separable metric spaces S of finite dimension, $m \leqq 2n + 1$, where m is the homeomorphic linear embedding characteristic, and n is the topological dimension. The question is raised whether a similar inequality holds for more general spaces S.

In section 2 we discuss the linear embedding characteristic, and give simple examples of continuous, non-homeomorphic injections. In section 3 the embedding by bounded functions is examined, and in section 4 it is proved that if a compact space S has finite simple dimension at each of its points, then it has finite linear embedding characteristic. Section 5 is devoted to observations about spaces S which are subsets of a Euclidean space.

A projected study by the writer, comparing and contrasting the Katětov dimension (which is defined in terms of the number of generators of the ring of bounded continuous functions on S – see [4], [10]) with the simple dimension, is unfinished at the time of this writing, and will be the subject of a later paper.

2. The linear embedding characteristic. For a Tikhonov space S, let us denote by $C(S)$ the ring of all real-valued continuous functions on S, and by $B(S)$ the subring of the bounded real continuous functions. Call an indexed subset $\{u_j\}$ of $C(S)$, where the functions in $\{u_j\}$ are assumed to be linearly independent on S, a *separating set* for S, in case the subset separates the points of S, i.e. in case for each pair s, s' of different points of S, there exists u_j in the subset such that $u_j(s) \neq u_j(s')$. We say that S has finite linear embedding characteristic $\leqq m$, if there is a separating set for S which has the finite set $J = \{1, \ldots, m\}$ as index, i.e. if a finite collection of functions $u_1, \ldots, u_m$ separates the points of S. For any $\{u_j\} \subset C(S)$ and index J, $\{u_j(s)\}$ defines a continuous evaluation map $\varphi = \varphi(\{u_j\})$ of S into the Cartesian product of real lines with index J. Evidently $\varphi(\{u_j\})$ is injective if and only if $\{u_j\}$ is a separating set for S.

Theorem 1. *If $\{u_j\}$, $\{v_j\}$, both with $J = \{1, \ldots, m\}$, are two separating sets for S, and if U, V are the respective images of S in Euclidean space E^m under the corresponding evaluation maps, then V is a non-singular linear transform of U whenever $\{u_j\}$, $\{v_j\}$ span the same m-dimensional subspace of $C(S)$.*

Proof. If $\langle\{u_j\}\rangle = \langle\{v_j\}\rangle$, then the v_j's must be linearly expressible in terms of the u_j's, with a non-singular matrix. Therefore V is a non-singular linear transform of U.

Example 1. In case S is the closed unit interval I, or the closed n-dimensional hypercube I^n, $n \leqq m$, then since S is compact, the injective evaluation map φ is a homeomorphism of S with its image U in E^m. But if S is the open unit interval or hypercube, the evaluation map may have no continuous extension, or no continuous injective extension, on the closed hypercube. In case of a continuous non-injective extension, the image U under the extended map of course is a Peano space.

Theorem 2. *If $v_1, \ldots, v_n$, not assumed linearly independent, separate the points of S, and if $\{u_1, \ldots, u_m\}$, $n < m$, is a separating set for S, then image V is the linear transform of image U by a linear transformation of rank n, whenever $\langle v_1, \ldots, v_n\rangle \subset \langle u_1, \ldots, u_m\rangle$ and $\langle v_1, \ldots, v_n\rangle$ is n-dimensional.*

The proof is similar to the proof of Theorem 1, and therefore is omitted.

It is easy to see that the converses of Theorems 1 and 2 are not true.

Definition. For a space S which has finite linear embedding characteristic, the *embedding characteristic* of S is m, if it is not $\leqq(m-1)$, but is $\leqq m$.

In terms of rectangular coordinates $v_1, \ldots, v_n$ in Euclidean space E^n, and $u_1, \ldots, u_m$ in Euclidean space E^m, an injective continuous mapping g of a subset $V \subset E^n$ onto a subset $U \subset E^m$ is given by m real functions $u_1 = g_1(v_1, \ldots, v_n), \ldots, u_m = g_m(v_1, \ldots, v_n)$. Through g, a continuous injection ψ of S onto V, corresponding to a separating set of n functions $v_1(s), \ldots, v_n(s)$ for S, determines a continuous injection $\varphi = g \circ \psi$ of S onto U in E^m, for which $u_1(s) = g_1(v_1(s), \ldots, v_n(s)), \ldots, u_m(s) = g_m(v_1(s), \ldots, v_n(s))$ form the corresponding separating set of m functions.

If the linear embedding characteristic of a space S is m, then not only is E^m the smallest flat containing the injective continuous image U, but also there does not exist a continuous injection of U into a lower-dimensional Euclidean space. For if there were such an injection, then there would be a separating set of continuous functions for S containing fewer than m functions.

Examples 2, 3, 4, 5. The half-open interval $0 \leqq v_1 < 1$ in E^1 has the circle in E^2 as an injective continuous, but non-homeomorphic, image. In E^2, consider the square $0 \leqq v_1 < 1$, $0 \leqq v_2 < 1$. A torus embedded in E^3 is the injective continuous, but non-homeomorphic, image of the square. In E^3, consider V consisting of the cylinder $v_1^2 + v_2^2 = 1$, $v_3 \geqq 0$, and the origin $(0, 0, 0)$. The non-compact space V has as a continuous injective image in E^2, the compact space U which consists of a closed unit disc. In E^2, let V consist of the open disc $v_1^2 + v_2^2 < 1$ and the point $(0, -1)$. The open disc, plus identified points $(0, -1)$ and $(0, 1)$, is a non-homeomorphic injective continuous image of V. The disc with identified points may be embedded homeomorphically as a subspace U of E^2.

Problem 1. Any Peano space is an injective continuous image of a subset of the closed unit interval. If V contains any n-ball, then by the Brouwer invariance of domain, a continuous injective g cannot lower the dimension of V. Does there exist a V not containing an n-ball, such that E^n is required for a homeomorphic embedding, but only E^m with $m < n$ in order to have an injective continuous image U of V?

For the homeomorphism g of the open disc in E^2 onto the open strip $-\infty < v_1 < \infty$, $0 < v_2 < 1$ in E^2, there is no continuous extension of g on the closed disc with range in E^2. For the square and injective mapping g to E^3 as in Example 3, g restricted to the open square is a homeomorphism, and has a continuous extension on the closed square, but no such extension which is injective on the closed square.

Problem 2. For an injective mapping g of $V \subset E^n$ onto $U \subset E^m$, does there always exist a dense subset V_1 of V, having the same topological dimension as V, such that g restricted to V_1 is a homeomorphism?

Theorem 4 below points out that an injective evaluation map φ of S is a homeomorphism iff, when $\varphi(S)$ is replaced by a homeomorph in the product of closed intervals with index J, there exists a compactification γS which is such that there exists a continuous and injective extension of φ on γS.

3. Linear embedding by bounded functions. Given any finite collection $\{u_1, \ldots, u_m\}$ of functions on S and the corresponding evaluation map φ, then the image $U = \varphi(S)$ in Euclidean space E^m is bounded iff the functions $u_1, \ldots, u_m$ are all bounded, i.e. iff $\langle u_1, \ldots, u_m \rangle$ is a subspace of $B(S)$.

Theorem 3. *For each finite collection $\{u_j\}$ of real continuous functions on S, and the corresponding evaluation map φ of S onto U in Euclidean space, there is a collection of bounded real continuous functions $\{v_j\}$ on S, and a corresponding* *evaluation map of S, onto a bounded subset V of Euclidean space which is homeomorphic with U. If $\langle\{u_j\}\rangle$ is of dimension m, then also $\langle\{v_j\}\rangle$ may be of dimension m.*

Proof. The proof is obtained by constructing a radial homeomorphism, using the function $r' = (2/\pi)\arctan r$. Details are left to the reader.

By Theorem 3, if S has finite linear embedding characteristic, then given a separating set $\{u_j\}$ for S, $J = \{1, \ldots, m\}$, we may assume that the functions $u_j(s)$ and the injective image U of S in E^m are bounded.

Theorem 4. *If $\{u_j\}$ with index J is a separating set for S, and if $\langle\{u_j\}\rangle \subset B(S)$, then the injective mapping φ of S onto U is a homeomorphism iff there exists a compactification γS of S, and an injective continuous extension of φ on γS, with range in the product with index J of real lines.*

Proof. Since the coordinate functions u_j are bounded, the image $U = \varphi(S)$ is included in the compact product of closed finite real intervals. Therefore the closure $\bar{U}$ of U is a compact Hausdorff space, and in case φ is a homeomorphism, we may take for γS a copy of $\bar{U}$ containing S as dense subspace; φ then may be extended to be a homeomorphism of γS onto $\bar{U}$.

In case there is a compactification γS and an injective continuous extension φ^- on γS of φ, then by Theorem 8 on p. 141 of [9], the mapping φ^- is a homeomorphism on γS, and in particular its restriction φ on S is a homeomorphism.

4. Simple dimension. We say that the *simple dimension of S at* a point $s \in S$ is $\leqq n$, in case there exists a set $\{u_j\} \subset C(S)$, $J = \{1, \ldots, n\}$, and an open neighbourhood O_s of s, such that $\{u_j(s)\}$ is a separating set for the points of O_s. The simple dimension at s is equal to n if it is $\leqq n$, but not $\leqq (n-1)$; and the *simple dimension of S* is n in case the simple dimension is equal to n for at least one point $s \in S$, and $\leqq n$ at all other points of S.

If a space S has finite linear embedding characteristic m, then of course it has simple dimension $\leqq m$. The next theorem shows conversely, at least in case S is compact, that a space S which has finite simple dimension must also have finite linear embedding characteristic. Any separable metric space S of topological dimension n has linear embedding characteristic $m \leqq 2n + 1$. (See page 60 of [7].)

Theorem 5. *If a space S is compact and has finite simple dimension at each of its points, then it has finite simple dimension, and finite linear embedding characteristic.*

Proof. Since a compact Hausdorff space is normal, we have that for each pair of points s, s' of S, $s \neq s'$, there are disjoint open neighborhoods O_s, $O_{s'}$ of respectively s, s', and a function from $C(S)$ or $B(S)$, which separates O_s, $O_{s'}$. For each point (s, s') of the Cartesian product $S \times S$, with $s \neq s'$, choose such a pair of neighborhoods; let $U(s, s')$ be their Cartesian product. The Cartesian products $O_s \times O_s$, where the neighborhoods O_s are chosen possessing the property mentioned in the definition of the simple dimension at s, cover the diagonal of $S \times S$. By compactness, a finite subset of the $\{O_s \times O_s\}$ and of the $\{U(s, s')\}$ cover $S \times S$. Thus only a finite number of subspaces from the collection of finite dimensional linear subspaces of $C(S)$ and only a finite number of basic functions, are required to separate the points of S.

5. Remarks on subsets of Euclidean space. Since Euclidean space has $\mathfrak{c}$ points, any space S having more than $\mathfrak{c}$ points, such as for example the Stone-Čech compactification βN of the discrete set of integers N, cannot be in one-to-one correspondence with any subset of Euclidean space, and in particular it cannot have finite linear embedding characteristic. Thus of course such a space S cannot be separable and metrizable.

Problem 3. Describe the spaces S having more than $\mathfrak{c}$ points, which have finite simple dimension, or finite topological dimension, with respect to properties including generalized separability, connectivity, metrizability.

Any subset S of Euclidean space E^n either is of topological dimension $\leqq n$, or is of infinite topological dimension. For as a subset of E^n, S is a separable metrizable space; if $S \subset E^n$ were of finite dimension $m > n$, since $E^n \subset E^m$, we may consider that $S \subset E^m$. Then by Theorem IV 3 on page 44 of [7], S must contain an open set of dimension m, but since $S \subset E^n$, this is impossible by the Brouwer invariance of domain. Compare this observation with Theorem 16.22 on page 251 of [4]: Any compact subset of E^n is of dimension $\leqq n$. Compare it also with the following well-known theorem (see [10]): A metrizable space S is of dimension $\leqq n$ iff for each $\varepsilon > 0$ there exists an ε-mapping of S onto a polyhedron of dimension $\leqq n$. The definition of topological dimension in [4] is slightly different than those of the covering and inductive dimensions, which coincide for separable metric spaces [10].

A space is realcompact or Hewitt-compact iff it is homeomorphic with a closed subset of a Cartesian product of real lines. Any subset of Euclidean space is realcompact [4]. Call a space *finite-realcompact* if the number of lines in the product may be finite. Euclidean space of dimension n is the Cartesian product of n lines, so closed subsets of E^n are finite-realcompact. Also many non-closed subsets of E^n, for example the hypercube $0 \leqq v_i < 1$, $i = 1, \ldots, n$, are homeomorphic with closed subspaces of Euclidean space. (The described hypercube is homeomorphic with the closed non-negative orthant of E^n.)

Problem 4. Characterize those subsets of Euclidean space which are not finite-realcompact, i.e. not homeomorphic with any closed subset of Euclidean space.

By Lemma 6.11 on page 92 of [4], the homeomorphism f of a non-finite-realcompact subset S of Euclidean space E^n, onto the closed subset of the infinite Cartesian product of lines, has no continuous extension on the closure $\bar{S}$ of S in E^n. (By the cited Lemma, if an extension F were continuous, the inverse image under F of the closed subset would be the non-closed subset S of E^n, contradicting continuity.) Each separable metric space of finite (or infinite) topological dimension has a separable metric compactification of the same dimension. We regard the number of lines required in the Cartesian product, for embedding S as a closed subset, as a sort of measure of the dimension of S; each non-finite-realcompact subset S of E^n has its closure $\bar{S}$ in E^n as a finite-realcompactification. (The finite-realcompactification reduces the measure of the dimension from infinite to finite.)

For a set S with topology $\mathscr{S}$, let us denote the embedding characteristic by emb $(S, \mathscr{S})$. Call a topology $\mathscr{U}$ *ultimate* in case for any larger topology $\mathscr{T}$, we have emb $(S, \mathscr{T}) <$ emb $(S, \mathscr{U})$. For a set S with at least two and not more than $\mathfrak{c}$ elements, the discrete topology is ultimate for the case emb $(S, \mathscr{S}) = 1$. (There exists a single real function which separates the points of S.) Similarly to Example 3, by cutting holes in a torus and retaining half of the boundaries, an injective pre-image of the torus is obtained. But the non-compact pre-image is homeomorphically embeddable in the plane, and thus has embedding characteristic 2.

Problem 5. For the case emb $(S, \mathscr{S}) = n > 1$, does there exist a topology $\mathscr{S}$ for which there is no ultimate topology $\mathscr{U}$? That is, is it possible to have $\mathscr{S}$ such that there exist an infinite chain of successively larger topologies, all with embedding characteristic n, which has no upper bound $\mathscr{V}$ with emb $(S, \mathscr{V}) =$ emb $(S, \mathscr{S}) = n$?

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